Attenuation of Uncertain Disturbances through Fast Control Inputs ⋆

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Abstract: We present a class of controls that provide an effect similar to the one produced by conventional matching conditions between control and disturbance, but now for a broader class of systems. This is the class of piecewise-constant functions with varying amplitudes, generated by approximations of "ideal controls" — linear combinations of delta-functions and their higher-order derivatives. Such a class allows to calculate feedback controls by solving problems of open-loop control.

Keywords: set-membership uncertainty, fast controls, disturbance attenuation

1. PROBLEM

Consider the following linear system with control u and uncertain disturbance (noise) v:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t), \quad t \in [t_0, t_1]. \quad (1)$$

The vector dimensions are $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^k$, $m, k \leq n$. The time interval $[t_0, t_1]$ is fixed in advance. The given matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{n\times k}$ are sufficiently smooth for our constructions.

Disturbance $v(t)$ is a piecewise-continuous function subject to hard bound $v(t) \in \mathscr{Q}(t)$, $t \in [t_0, t_1]$, where $\mathscr{Q}(t)$ is a set-valued function with values in $\operatorname{conv}\mathbb{R}^k$ — the class of non-empty convex compacts in $\mathbb{R}^k$. The function $\mathscr{Q}(t)$ is continuous in the Hausdorff metric. It could be, for example, defined by inequalities $|v_i(t)| \leq \nu_i$, $i = 1, \dots, k$.

The aim of the control is to steer the system to a given target set $\mathscr{M} \in \operatorname{conv}\mathbb{R}^n$ at a prescribed time t_1, despite the disturbance.

Let us describe the considered classes of control functions. It is well known from the theories of control under uncertainty and differential games (see Leitmann (1982); Başar and Bernhard (1995); Krasovski (1971); Krasovski and Subbotin (1988); Kurzhanski (1999)) that if $B(t) \equiv C(t)$ and control $u(t)$ belongs to the same class as $v(t)$, namely, $u(t) \in \mathscr{P}(t)$, where $\mathscr{P}(t) = \alpha(t)\mathscr{Q}(t)$, $|\alpha(t)| \geq 1$, so that the bounds on u and v are similar ("homothetic"), then the solutions to the corresponding min-max terminal control problems (like min-max over u, v of the terminal distance to the target set $\mathscr{M} \in \operatorname{conv}\mathbb{R}^n$) have the following property: the solution in the classes of open-loop and closed-loop controls coincide. A completely different situation arises when the mentioned similarity condition does not hold. In this case the closed-loop control problem is much harder to solve than for the open-loop, and may require a significant increase in computational burden.

Problem 1. Specify classes of controls that allow to reduce the closed-loop terminal min-max problem to open-loop.

Here we present the class of controls that provide an effect similar to the one produced by conventional matching conditions between u and v, but now for a broader class of systems. This is the class of piecewise-constant functions with varying amplitudes, generated by approximations of "ideal controls" — linear combinations of delta-functions and their higher-order derivatives. Such a class allows to calculate feedback controls by solving problems of open-loop control.

2. GENERALIZED CONTROLS

Let the control input be a generalized function (a distribution) of order s. The latter may be presented as the sum of generalized derivatives of functions of bounded variation (see Gelfand and Shilov (1964); Schwartz (1950)):

$$u(t) = \sum_{j=0}^{s} \frac{d^{j+1}U_j(t)}{dt^{j+1}}, \quad U_j(\cdot) \in BV([t_0, t_1]; \mathbb{R}^m). \quad (2)$$

In particular, as indicated by Kurzhanski and Osipov (1969), the optimal generalized control problem of steering the system to a prescribed state *in the absence of uncertainty* has the form

$$u(t) = \sum_{i=1}^{n}\sum_{j=0}^{s} h_{i,j}\delta^{(j)}(t - \tau_j), \quad (3)$$

where $\delta(t) = \chi'(t)$ is the delta-function — the generalized derivative of the Heaviside function $\chi(t) \in BV[t_0, t_1]$; vectors $h_{i,j} \in \mathbb{R}^m$ define the direction and the amplitude of the generalized impulses, τ_i are the times of these impulses.

Substituting control input (2) into the original differential equation (1), we come to the following impulse control system (see Kurzhanski and Osipov (1969)):

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$$dx(t) = A(t)x(t)dx(t) + \mathscr{B}(t)dU(t) + C(t)v(t)dt, \quad (4)$$

on $t \in [t_0, t_1]$, where $\mathscr{B}(t) = [L_0(t) \cdots L_s(t)]$, $U(t) = [U_0(t) \cdots U_s(t)] \in BV([t_0, t_1]; \mathbb{R}^{m(s+1)})$ is an impulse control. The aim of the control is to ensure $x(t_1 + 0) \in \mathscr{M}$. Matrix functions $L_j(t)$ are here defined by the recurrence relations

$$L_0(t) = B(t), \quad L_j(t) = A(t)L_{j-1}(t) - L'_{j-1}(t). \quad (5)$$

Therefore, higher-order generalized impulses may increase the control possibilities in the sense that $\operatorname{Range}\mathscr{B}(t) \supseteq \operatorname{Range} B(t)$ (here and further Range is the column space of a matrix).

Assumption 1. There exists an $s \le n - 1$ such that $\operatorname{Range}\mathscr{B}(t) \supseteq \operatorname{Range} C(t)$ for all $t \in [t_0, t_1]$.

This assumption holds if, for example, $A(t) \equiv A$, $B(t) \equiv B$, and $[A, B]$ is a controllable pair. In this case the minimum value of s coincides with the controllability index of the system.

We now we replace the "ideal" impulse control in system (4) by physically realizable bounded functions. To do that we introduce a hard bound on the control input $\mathbf{u}(t) = dU/dt$: $\mathbf{u}(t) \in \mathscr{P}(t)$. Then system (4) acquires the form

$$\dot{x}(t) = A(t)x(t) + \mathscr{B}(t)\mathbf{u}(t) + C(t)v(t), \quad t \in [t_0, t_1]. \quad (6)$$

Here $\mathbf{u}(t) = [u_0(t) \cdots u_s(t)] \in \mathbb{R}^{m(s+1)}$, and the aim of the control is again $x(t_1) \in \mathscr{M}$.

It is known that if the *matching condition* holds [1]

$$(\mathscr{B}(t)\mathscr{P}(t) \dot{-} C(t)\mathscr{Q}(t)) + C(t)\mathscr{Q}(t) = \mathscr{B}(t)\mathscr{P}(t) \quad (7)$$

then the solution of feedback control problem simplifies significantly (see Krasovski and Subbotin (1988); Kurzhanski (1999)). This condition is equivalent to convexity of $f(\ell) = \rho(\ell \mid \mathscr{B}(t)\mathscr{P}(t)) - \rho(\ell \mid C(t)\mathscr{Q}(t))$ — the difference of support functions for sets $\mathscr{B}(t)\mathscr{P}(t)$ and $C(t)\mathscr{Q}(t)$.

Our aim will be to match the bounds of control and disturbance in order to satisfy condition (7). With set $\mathscr{Q}(t)$ given, there exist at least the next two approaches:

(1) Choose an appropriate $\mathscr{P}(t)$.
(2) Choose $\mathscr{P}(t)$ such that $\mathscr{B}(t)\mathscr{P}(t) \dot{-} C(t)\mathscr{Q}(t) \neq \emptyset$. Then choose a set $\hat{\mathscr{Q}}(t) \supseteq \mathscr{Q}(t)$ such that the matching condition will hold.

2.1 Example

Consider a three-body oscillating system (see Vostrikov et al. (2006))

$$\begin{cases} m\ddot{w}_1 = k(w_2 - 2w_1) + mv_1(t), \\ m\ddot{w}_2 = k(w_3 - 2w_2 + w_1) + mv_2(t), \\ m\ddot{w}_3 = k(w_2 - w_3) + mu(t) + mv_3(t), \end{cases} \quad (8)$$

that consists of a chain of linked weights of mass m connected by springs of stiffness k. Variables w_j are the displacements of the weights from equilibrium. Control u and disturbance v_j are the forces applied to the weights. We assume the hard bound on disturbance $|v_j(t)| \le \nu_j$, $j = \overline{1,3}$.

[1] Symbol $\dot{-}$ denotes the geometric (Minkowski) difference of the sets: $A \dot{-} B = \{x \mid x + B \subseteq A\}$.

The matching condition for the system (8) does not hold since the control u only enters the last equation, whereas the disturbance is present in each of the equations.

Rewriting the system (8) in normal form (denoting $\omega = k/m$) we get

$$\begin{cases} \dot{x}_j = x_{3+j}, \quad j = \overline{1,3}; \\ \dot{x}_4 = \omega(x_2 - 2x_1) + v_1(t), \\ \dot{x}_5 = \omega(x_3 - 2x_2 + x_1) + v_2(t), \\ \dot{x}_6 = \omega(x_2 - x_3) + u(t) + v_3(t). \end{cases} \quad (9)$$

To fulfill condition $\operatorname{Range}\mathscr{B}(t) \supseteq \operatorname{Range} C(t)$ it is necessary to apply distributions at least of order $s \ge 4$. In our example we choose $s = 5$. Then matrix $\mathscr{B}(t)$ will be

$$\mathscr{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & \omega & 0 & -3\omega^2 \\ 0 & 1 & 0 & -\omega & 0 & 2\omega^2 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & \omega & 0 & -3\omega^2 & 0 \\ 1 & 0 & -\omega & 0 & 2\omega^2 & 0 \end{bmatrix}.$$

To match the bounds on u and v we first perform a linear substitution of variables:

$$\hat{u}_1 = u_1 - \omega u_3 + 2\omega^2 u_5,\ \hat{u}_3 = \omega u_3 - 3\omega^2 u_5,\ \hat{u}_3 = \omega^2 u_5,$$
$$\hat{u}_2 = u_2 - \omega u_4 + 2\omega^2 u_6,\ \hat{u}_4 = \omega u_4 - 3\omega^2 u_6,\ \hat{u}_6 = \omega^2 u_6.$$

Then the system (9) takes form

$$\begin{cases} \dot{x}_1 = x_4 + \hat{u}_6(t),\ \dot{x}_4 = \omega(x_2 - 2x_1) + +\hat{u}_5(t) + v_1(t), \\ \dot{x}_2 = x_5 + \hat{u}_4(t),\ \dot{x}_5 = \omega(x_3 - 2x_2 + x_1) + \hat{u}_3(t) + v_2(t), \\ \dot{x}_3 = x_6 + \hat{u}_2(t),\ \dot{x}_6 = \omega(x_2 - x_3) + \hat{u}_1(t) + v_3(t). \end{cases}$$

We further choose the bounds on the controls as

$$|\hat{u}_1(t)| \le \alpha_3\nu_3,\ |\hat{u}_3(t)| \le \alpha_2\nu_2,\ |\hat{u}_5(t)| \le \alpha_1\nu_1,\ \alpha_j \ge 1.$$

Controls $\hat{u}_2(t)$, $\hat{u}_4(t)$, $\hat{u}_6(t)$ may be bounded by an arbitrary convex set. In particular, we may set $\hat{u}_2(t) = \hat{u}_4(t) = \hat{u}_6(t) = 0$ in order to preserve the original physical sense (the control is a force which acts only on the velocities, but not on the displacements).

3. CONTROL INPUTS FOR THE ORIGINAL SYSTEM

The suggested approach allows us to find a feedback control for system (6), so that then, for a certain realization of $v(t)$, one may calculate the control trajectory $\mathbf{u}(t)$. After that it is necessary to indicate the corresponding control input for the original system (1).

It is not possible to apply representation (2) directly, since the smoothness (and even the continuity) of function $\mathbf{u}(t)$ is originally not guaranteed. To overcome this difficulty, we suggest to approximate the generalized controls using one of the following schemes.

(1) In (3), replace the derivatives of delta-functions by their bounded approximations. In this case we come to a system different from (6), for which it is necessary to apply the theory of the above.
(2) Solve the control problem for the system (6), then approximate the realization of the control $\mathbf{u}(t)$ by functions sufficiently smooth to apply the representation (2).

3.1 First scheme

Following (Dar'in and Kurzhanskii (2007); Kurzhanski and Daryin (2008)), we replace in (3) the derivatives of the delta-function by their piecewise-constant approximations:

$$u(t) = \sum_{i=1}^{n} \sum_{j=0}^{s} h_{i,j} \Delta_h^{(j)}(t - \tau_j), \tag{10}$$

where $\Delta_h^{(0)}(t) = h^{-1}\mathbf{1}_{[0,h]}(t)$,

$$\Delta_h^{(j)}(t) = h^{-1}\left(\Delta_h^{(j-1)}(t) - \Delta_h^{(j-1)}(t-h)\right). \tag{11}$$

Note the following properties of the these approximations.

1. The weak* limit (as $h \to 0$) of $\Delta_h^{(j)}(t)$ in the space of generalized functions of order j is $\delta^{(j)}(t)$.
2. Recurrence relations (5) lead to the next explicit form of these functions:

$$\Delta_h^{(j)}(t) = h^{-(j+1)} \sum_{i=0}^{j} (-1)^i C_j^i \mathbf{1}_{[ih,(i+1)h]}(t).$$

The Cauchy formula for system (6) is

$$x(\vartheta) = X(\vartheta, t_0)x_0 + \sum_{j=0}^{s} \int_{t_0}^{\vartheta} X(\vartheta, t) L_j(t) u_j(t) dt + \int_{t_0}^{\vartheta} X(\vartheta, t) C(t) v(t) dt. \tag{12}$$

Note that functions $L_j(t)$ from (5) are defined by relations

$$L_j(t) = (-1)^j X(t, t_0)[X(t_0, t)B(t)]^{(j)}.$$

We then represent these as convolutions with derivatives of the delta-function:

$$L_j(t) = X(t, t_0) \int_{\mathbb{R}} X(t_0, \tau) B(\tau) \delta^{(j)}(\tau - t) d\tau,$$

After that we pass to approximations (11):

$$M_h^{(j)}(t) = \int_t^{t+(j+1)h} X(t, \tau) B(\tau) \Delta_h^{(j)}(\tau - t) d\tau. \tag{13}$$

Theorem 1. Matrix functions $\mathscr{M}_h^{(j)}(t)$ satisfy recurrence relations

$$M_h^{(j)}(t) = h^{-1}(M_h^{(j-1)}(t) - X(t, t+h) M_h^{(j-1)}(t+h)),$$

$$M_h^{(0)}(t) = h^{-1} \int_t^{t+h} X(t, \tau) B(\tau) d\tau.$$

In particular, for $A(t) \equiv A$, $B(t) \equiv B$

$$M_h^{(j)} = h^{-j}(I - e^{-Ah})^j M_h^{(0)}, \quad M_h^{(0)} = h^{-1}\left[\int_0^h e^{At} dt\right] B.$$

Theorem 2. Let the matrix function $A(t)$ be continuous, and $B(t)$ be $s+1$ times continuously differentiable. Then functions $M_h^{(j)}(t)$ will converge to $L_j(t)$ uniformly on $[t_0, t_1]$, $j = 0, \ldots, s$ as $h \to 0$.

Corollary 1. Under the stated conditions, the matrix function $\mathscr{M}_h(t) = \left(M_h^{(0)}(t) \cdots M_h^{(s)}(t)\right)$ converges to $\mathscr{B}(t)$ uniformly on $[t_0, t_1]$ as $h \to 0$.

Corollary 2. If $\operatorname{rank}\mathscr{B}(t) \equiv n$, then for sufficiently small $h > 0$ one also has $\operatorname{rank}\mathscr{M}_h(t) \equiv n$.

Substituting in (12) the functions $L_j(t)$ by $M_h^{(j)}(t)$ we get

$$x_h(\vartheta) = X(\vartheta, t_0)x_0 + \sum_{j=0}^{s} \int_{t_0}^{\vartheta} X(\vartheta, t) M_h^{(j)}(t) u_j(t) dt + \int_{t_0}^{\vartheta} X(\vartheta, t) C(t) v(t) dt. \tag{14}$$

This is the Cauchy formula for system

$$\dot{x}_h(t) = A(t)x_h(t) + \mathscr{M}_h(t)\mathbf{u}(t) + C(t)v(t). \tag{15}$$

Theorem 3. Trajectories $x_h(t)$ of system (15) converge uniformly to the trajectory $x(t)$ of the system (6) with $h \to 0$ over $[t_0, t_1]$.

Theorem 4. Let $u(t) \equiv 0$, $v(t) \equiv 0$ for $t \in (\vartheta, \vartheta + (s+1)h]$. Then $x_h(\vartheta + (s+1)h) = x(\vartheta + (s+1)h)$, where $x(t)$ is the trajectory of the original system (1) with control

$$u_h(t) = \sum_{j=0}^{s} \int_{t_0}^{t} \Delta_h^{(j)}(t - \tau) u_j(\tau) d\tau. \tag{16}$$

Note that $u_h(t)$ is non-anticipative: it depends only on values of $\mathbf{u}(\tau)$ for $\tau \leq t$, i.e. it may be calculated using only the information available by time t.

Theorems stated above provide the following scheme for calculating control inputs in the original system.

1. Fix $h > 0$ and consider system (15).
2. Apply one of the approaches to choose the bounds on control and disturbance, with $\mathscr{B}(t)$ replaced by $\mathscr{M}_h(t)$. (Due to corollary 2, if $\operatorname{Range}\mathscr{B}(t) = \mathbb{R}^n$, then $\operatorname{Range}\mathscr{M}_h(t) = \operatorname{Range}\mathscr{B}(t)$.)
3. For system (15) with chosen constraints design a feedback control $\mathscr{U}(t, x)$.
4. Find the realization of control trajectory $\mathbf{u}(t)$.
5. Using (16), find the control input for the original system (1). (Since $u_h(t)$ depends only on the past values of $\mathbf{u}(t)$, it may be calculated on-line.)

3.2 Second scheme

Here we briefly describe the second scheme of calculating the control input for the original system. Let $\mathbf{u}(t) = [u_0(t) \cdots u_s(t)]$ be the realization of the control of system (6). We approximate it by convolving with sufficiently smooth functions $\hat{\mathbf{u}}(t) = [\hat{u}_0(t) \cdots \hat{u}_s(t)]$:

$$\hat{u}_j(t) = h^{-1} \int_{t_0}^{t_1} K_j((t - \tau)/h) u_j(\tau) d\tau.$$

The convolution kernels $K_j(t)$ should satisfy the following requirements: $K_j(t) = 0$ for $t < 0$; $K_j(t) \geq 0$ for $t \geq 0$; $K_j(t)$ is j times continuously differentiable; they satisfy the normalization condition: $\int_0^\infty K_j(t) dt = 1$.

One may select $K_j(t)$, for example, as the following piecewise-polynomial functions:

$$K_j(t) = \mathbf{1}_{[0,1]}(t) C_j (t(1-t))^{j+1}, \quad C_j = \frac{(2j+3)!}{((j+1)!)^2}.$$

The control $\hat{\mathbf{u}}(t)$ corresponds to the next control input for the original system (1):

$$\hat{u}(t) = \sum_{j=0}^{s} \hat{u}_j^{(j)}(t) = \sum_{j=0}^{s} h^{-(j+1)} \int_{t_0}^{t_1} K_j^{(j)}((t-\tau)/h) u_j(\tau) d\tau.$$

This approximation has the following properties.

1. $\hat{u}_j(t) \to u_j(t)$ almost everywhere when $h \to 0$.

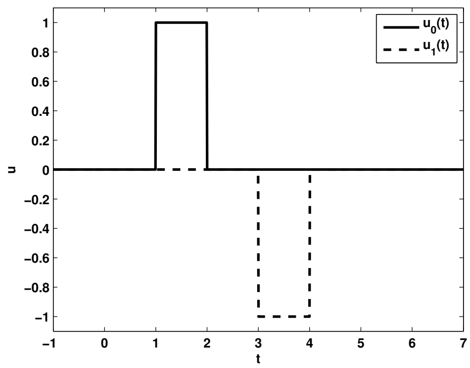

Fig. 1. Control input $u_j(t)$ for the system (6)

(2) Trajectories $\hat{x}(t)$ of the system (1) under control $\hat{u}(t)$ coincide with the trajectories of the system (6) under control $\hat{\mathbf{u}}(t)$. The latter converge pointwise to the trajectories $x(t)$ of the system (6) under control $\mathbf{u}(t)$.
(3) $\hat{u}(t)$ depends only on values of $\mathbf{u}(\tau)$ for $\tau \leq t$, i.e. it may be calculated using only the information available by time t.

3.3 Example

Consider system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + v_1(t), \\ \dot{x}_2(t) = u(t) + v_2(t), \end{cases}$$

with hard bound on disturbance as $|v_1| \leq \mu_1$, $|v_2| \leq \mu_2$.

For this system we have

$$\mathscr{B}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{M}_h(t) = \begin{bmatrix} h/2 & 1 \\ 1 & 0 \end{bmatrix}.$$

To apply the first scheme, we make a linear change of variables: $\hat{u}_1(t) = hu_1(t)/2 + u_2(t)$, $\hat{u}_2(t) = u_1(t)$, which leads to system (15) of form

$$\begin{cases} \dot{x}_{h1}(t) = x_{h2}(t) + \hat{u}_1(t) + v_1(t), \\ \dot{x}_{h2}(t) = \hat{u}_2(t) + v_2(t). \end{cases}$$

Here one may choose the following constraint on control: $|\hat{u}_1| \leq \nu_1$, $|\hat{u}_2| \leq \nu_2$, where $\nu_j \geq \mu_j$.

Applying the second scheme we get a system (6) of form

$$\begin{cases} \dot{x}_1(t) = x_2(t) + u_1(t) + v_1(t), \\ \dot{x}_2(t) = u_2(t) + v_2(t), \end{cases}$$

where the hard bound on control may be also chosen as $|u_1| \leq \nu_1$, $|u_2| \leq \nu_2$, with $\nu_j \geq \mu_j$.

Suppose that a control realization for system (6) is depicted in Fig. 1. Then Fig. 2 shows the control inputs for the original system (1) as calculated using both schemes. Here $t_0 = 0$, $t_1 = 5$, $h = 0{,}5$.

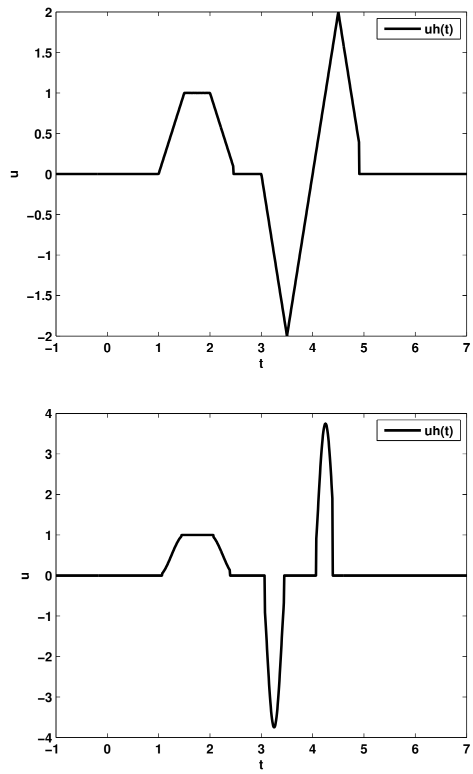

Fig. 2. Control input for the original system (1) calculated by the first (above) and the second scheme (below)

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