Attenuation of Uncertain Disturbances through Fast Control Inputs

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Control System

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t) \quad t \in [t_0, t_1]$$

- $x \in \mathbb{R}^n$ — position
- $u \in \mathbb{R}^m$ — control
- $v \in \mathbb{R}^k$ — disturbance (unknown)
- $[t_0, t_1]$ — fixed time interval
- $x(t_0) = x_0$ — initial position (known)
- Constraints:
  - $u(t) \in \mathscr{P}(t)$
  - $v(t) \in \mathscr{Q}(t)$
- Control objective: $x(t_1) \in \mathscr{M}$

Solution of the Feedback Control Problem is based on
L. S. Pontryagin's Alternated Integral

Max-min solvability (backward reachability) set:

$$W^{+}(t; t_1, \mathscr{M}) = \{x \in \mathbb{R}^n \mid \forall v(\cdot) \exists u(\cdot) : x(t_1; t, x, u(\cdot), v(\cdot)) \in \mathscr{M}\}$$

Alternated sums:

$$\mathscr{I}^{+}_{\mathscr{T}}[t] = W^{+}(t; \tau_1, W^{+}(\tau_1; \tau_2, \ldots W^{+}(\tau_k; t_1, \mathscr{M}) \ldots))$$

Alternated Integral:

$$\mathscr{I}[t] = \bigcap_{\mathscr{T}} \mathscr{I}^{+}_{\mathscr{T}}[t]$$

- Support function of $W^+[t]$ (for $A(t) \equiv 0$):

$$\rho\left(\ell \mid W^+[t]\right) = \text{conv}\Big\{\rho\left(\ell \mid \mathscr{M}\right) + \\ + \int_t^{t_1} \rho\left(-\ell \mid B(t)\mathscr{P}(t)\right) dt - \int_t^{t_1} \rho\left(\ell \mid C(t)\mathscr{Q}(t)\right) dt\Big\}$$

- Convex hull is calculated at each step.

- Simple solution for particular cases.
- **Matching Condition:** $\rho(-\ell \mid B(t)\mathscr{P}(t)) - \rho(\ell \mid C(t)\mathscr{Q}(t))$ is convex
- Homothety: $\mathscr{P}(t) = \alpha(t)\mathscr{Q}(t)$, $|\alpha(t)| \geq 1$
- Convex hull may be omitted:

$$\rho(\ell \mid W^{+}[t]) = \sout{\mathrm{conv}}\Big\{\rho(\ell \mid \mathscr{M}) + \\ + \int_{t}^{t_1} \rho(-\ell \mid B(t)\mathscr{P}(t))\,dt - \int_{t}^{t_1} \rho(\ell \mid C(t)\mathscr{Q}(t))\,dt\Big\}$$

- **Solution reduces to open-loop constructs:**

$$\mathscr{I}[t] = W^{+}[t].$$

Examples:

- Homothety ($\Rightarrow$ Matching Condition) **holds** for $\mu \geq \nu$:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t) + v(t), \end{cases} \qquad |u| \leq \mu,\ |v| \leq \nu.$$

- Matching Condition **does not hold** for any bound on u:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + v_1(t), \\ \dot{x}_2(t) = u(t) + v_2(t). \end{cases}$$

Problem

Problem

Indicate a class of control inputs that allows to solve the feedback control problem via open-loop constructs, for control systems lacking the Matching Condition.

Fast Controls

- Generalized control (for $t_0 = t_1$):

$$u(t) = \sum_{j=0}^{s} h_j \delta^{(j)}(t - t_1), \quad h_j \in \mathbb{R}^m$$

  - For large s, steers a completely controllable system from x_0 to x_1 **in zero time**.
- Fast Controls are **bounded approximations** of generalized controls.
  - Steer a completely controllable system from x_0 to x_1 **in arbitrarily small time**.
- **Fast Controls widen the class of problems solvable by open-loop constructs.**

Solution Scheme

Original System (1)

$\Downarrow$ (formal transition)

System with Generalized Controls (2)

$\Downarrow$ (reduction)

System with Impulse Controls (3)

$\Downarrow$ (additional bound on control)

System with Fast Controls (4)

$\Downarrow$ (solving the Feedback Control Problem with open-loop constructs)

Bounded control for system (4)

$\Downarrow$ (approximation)

Bounded control for system (1)

Generalized Controls

General Form

$$u(t) = \sum_{j=0}^{s} \frac{d^{j+1} U_j(t)}{dt^{j+1}}, \quad U_j(\cdot) \in BV([t_0, t_1]; \mathbb{R}^m)$$

In particular, in the absence of disturbances

$$u(t) = \sum_{i=1}^{n} \sum_{j=0}^{s} h_{i,j} \delta^{(j)}(t - \tau_j),$$

- $\delta(t) = \chi'(t)$
- $h_{i,j} \in \mathbb{R}^m$ — direction and amplitude of generalized impulses
- τ_i — timings of these impulses

System with generalized control inputs reduces to

$$dx(t) = A(t)x(t)dx(t) + \mathscr{B}(t)dU(t) + C(t)v(t)dt$$

- $U(t) = \begin{bmatrix} U_0(t) & \cdots & U_s(t) \end{bmatrix} \in BV([t_0, t_1]; \mathbb{R}^{m(s+1)})$ — Impulse Control
- $\mathscr{B}(t) = \begin{bmatrix} L_0(t) & \cdots & L_s(t) \end{bmatrix}$

$$L_0(t) = B(t), \quad L_j(t) = A(t)L_{j-1}(t) - \frac{dL_{j-1}(t)}{dt}, \quad j = \overline{1, s}.$$

- control objective is $x(t_1 + 0) \in \mathscr{M}$

Range $\mathscr{B}(t) \supseteq$ Range $B(t)$ — greater control capabilities

Assumption

There exists $s \leq n - 1$, for which Range $\mathscr{B}(t) \supseteq$ Range $C(t)$ for all $t \in [t_0, t_1]$.

Examples:

- $A(t) \equiv A$
- $B(t) \equiv B$
- $[A, B]$ is a controllable pair

System with Bounded Controls

Additional bound on control:

$$\mathbf{u}(t) = dU/dt \in \mathscr{P}(t).$$

$$\dot{x}(t) = A(t)x(t) + \mathscr{B}(t)\mathbf{u}(t) + C(t)v(t), \quad t \in [t_0, t_1]. \qquad (1)$$

- $\mathbf{u}(t) = \begin{bmatrix} u_0(t) & \cdots & u_s(t) \end{bmatrix} \in \mathbb{R}^{m(s+1)}$
- control objective is $x(t_1) \in \mathscr{M}$

Choosing Bounds on Control and Disturbance

Objective: balance bounds on control and disturbances to provide the **Matching Condition**:

$$(\mathscr{B}\mathscr{P} \dot{-} C\mathscr{Q}) + C\mathscr{Q} = \mathscr{B}\mathscr{P} \qquad (*)$$

Possible Approaches:

1. Either choose $\mathscr{P}$ from $(*)$
2. or:
   1. Choose $\mathscr{P}$ s.t. $\mathscr{B}\mathscr{P} \dot{-} C\mathscr{Q} \neq \emptyset$
   2. Choose $\hat{\mathscr{Q}} \supseteq \mathscr{Q}$ from $(*)$

Lemma

For any set $\mathscr{N}(t) \in \operatorname{conv} \operatorname{Range} \mathscr{B}(t)$ *there exists a set* $\mathscr{P}(t) \in \operatorname{conv} \mathscr{R}^{m(s+1)}$, *such that* $\mathscr{B}(t)\mathscr{P}(t) = \mathscr{N}(t)$.

Let

$$\mathscr{N}(t) = \alpha C(t)\mathscr{Q}(t) + \mathscr{N}_0(t), \quad \alpha \geq 1,$$

where $\mathscr{N}_0(t) \in \operatorname{conv} \operatorname{Range} \mathscr{B}(t)$ is an arbitrary set.

Suppose $A(t) \equiv A$, $B(t) \equiv B$, $[A, B]$ is a controllable pair:

$$\mathscr{B}(t) \equiv \mathscr{B} = \begin{bmatrix} B & AB & \cdots & A^s B \end{bmatrix}.$$

The disturbance $\mathbf{v}(t) = \begin{bmatrix} v_1(t) & \cdots & v_r(t) \end{bmatrix}$

$$C = \begin{bmatrix} k_1 A^{j_1} B & \cdots & k_r A^{j_r} B \end{bmatrix}, \quad 0 \le j_1 < j_2 < \cdots < j_r \le s; \quad k_i \in \mathbb{R}.$$

If the hard bound on disturbance $\mathscr{Q}$ is

$$\|v_1(t)\| \le 1, \quad \ldots, \quad \|v_s(t)\| \le 1,$$

then the hard bound on control may be chosen as

$$\|u_0(t)\| \le \mu_0, \quad \ldots, \quad \|u_s(t)\| \le \mu_s.$$

The Matching Condition holds when

$$\mu_{j_1} \ge k_1, \quad \ldots, \quad \mu_{j_r} \ge k_r.$$

Choosing Bound on Disturbance

Definition (Polovinkin, Balashov)

$\mathscr{X} \in \operatorname{conv}\mathbb{R}^n$ is a **generating set**, if $\forall \mathscr{Y} \subseteq \mathbb{R}^n$, s.t. $\mathscr{X} \stackrel{\cdot}{-} \mathscr{Y} \neq \emptyset$, $\exists \mathscr{Z} \in \operatorname{conv}\mathbb{R}^n$, s.t. $\mathscr{X} \stackrel{\cdot}{-} \mathscr{Y} + \mathscr{Z} = \mathscr{X}$

Note that

- $\mathscr{Z} \supseteq \mathscr{Y}$
- for $\mathscr{X}$ and $\mathscr{Z}$ the Matching Condition holds

Suppose $\mathscr{P}(t)$ satisfies the conditions:

1. $\mathscr{B}(t)\mathscr{P}(t) \stackrel{\cdot}{-} C(t)\mathscr{Q}(t) \neq \emptyset$;
2. $\mathscr{B}(t)\mathscr{P}(t)$ is a generating set in $\operatorname{Range}\mathscr{B}(t)$.

Replace the set $\mathscr{Q}(t)$ with a larger $\hat{\mathscr{Q}}(t) \supseteq \mathscr{Q}(t)$, so that the Matching Condition holds.

Choosing Bound on Disturbance

$n = 2$: any closed convex set is a generating set.
$n \geq 3$. Let $\mathscr{B}(t) = \begin{bmatrix} \mathscr{B}_1(t) & \mathscr{B}_2(t) \end{bmatrix}$, where $\mathscr{B}_1(t) \in \mathbb{R}^{n \times q}$ has full column rank. Respectively, $\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_1(t) & \mathbf{u}_2(t) \end{bmatrix}$, $\mathbf{u}_1(t) \in \mathbb{R}^q$.
Define bound on control $\mathbf{u}(t)$:

1. bound of $\mathbf{u_1}$ is
    1. either $|(\mathbf{u}_1)_j| \leq \mu_j$
    2. or $\|\mathbf{u}_1\| \leq \mu$.
2. $\mathbf{u_2} \in \mathscr{P}_2(t)$.

Then $\mathscr{B}(t)\mathscr{P}(t)$ will be a generating set. Numbers μ_j (μ) are chosen s.t. $\mathscr{B}_1(t)\mathscr{P}_1(t) \dot{-} C(t)\mathscr{Q}(t) \neq \emptyset$.

Calculating the Control Inputs

Recall:

$$u(t) = \sum_{j=0}^{s} \frac{d^{j+1} U_j(t)}{dt^{j+1}}, \quad U_j(\cdot) \in BV([t_0, t_1]; \mathbb{R}^m)$$

$$u(t) = \sum_{i=1}^{n} \sum_{j=0}^{s} h_{i,j} \delta^{(j)}(t - \tau_j),$$

? — the realization of control $\mathbf{u}(t)$ is not smooth

Possible approaches:

- replace δ-functions with approximations
- approximate $\mathbf{u}(t)$ by smooth functions

Calculating the Control Inputs
First Approach

$$u(t) = \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=0}^{s} h_{i,j} \Delta_h^{(j)}(t - \tau_j),$$

$$\Delta_h^{(0)}(t) = h^{-1} \mathbf{1}_{[0,h]}(t), \quad \Delta_h^{(j)}(t) = h^{-1} \left(\Delta_h^{(j-1)}(t) - \Delta_h^{(j-1)}(t-h) \right)$$

Matrix $\mathscr{B}(t)$ is replaced with

$$\mathscr{M}_h(t) = \begin{pmatrix} M_h^{(0)}(t) & \cdots & M_h^{(s)}(t) \end{pmatrix}$$

$$M_h^{(j)} = h^{-j}(I - e^{-Ah})^j M_h^{(0)}, \quad M_h^{(0)} = h^{-1} \left[\int_0^h e^{At} dt \right] B.$$

Theorem

Let $A(t) \in C[t_0, t_1]$, $B(t) \in C^{s+1}[t_0, t_1]$.
Then $M_h(t) \rightrightarrows \mathscr{B}(t)$ with $h \to 0$ uniformly on $[t_0, t_1]$.

Corollary: if $\operatorname{rank} \mathscr{B}(t) \equiv n$, then for $h \to 0$ we have $\operatorname{rank} \mathscr{M}_h(t) \equiv n$.

New system

$$\dot{x}_h(t) = A(t)x_h(t) + \mathscr{M}_h(t)\mathbf{u}(t) + C(t)v(t) \tag{2}$$

Theorem

Let $u(t) \equiv 0$, $v(t) \equiv 0$ *for* $t \in (\vartheta, \vartheta + (s+1)h]$. *Then* $x_h(\vartheta + (s+1)h) = x(\vartheta + (s+1)h)$, *where* $x(t)$ *is the trajectory of the original system under control*

$$u_h(t) = \sum_{j=0}^{s} \int_{t_0}^{t} \Delta_h^{(j)}(t-\tau)u_j(\tau)d\tau.$$

Approximate components of $\mathbf{u}(t)$ by smooth functions:

$$\hat{\mathbf{u}}(t) = \begin{bmatrix} \hat{u}_0(t) & \cdots & \hat{u}_s(t) \end{bmatrix}$$

$$\hat{u}_j(t) = h^{-1} \int_{t_0}^{t_1} K_j((t-\tau)/h) u_j(\tau) d\tau.$$

$$K_j(t) = \mathbf{1}_{[0,1]}(t) C_j (t(1-t))^{j+1}, \quad C_j = \frac{(2j+3)!}{((j+1)!)^2}.$$

Control input for the original system:

$$\hat{u}(t) = \sum_{j=0}^{s} \hat{u}_j^{(j)}(t) = \sum_{j=0}^{s} h^{-(j+1)} \int_{t_0}^{t_1} K_j^{(j)}((t-\tau)/h) u_j(\tau) d\tau.$$

Example

Consider a system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + v_1(t), \\ \dot{x}_2(t) = u(t) + v_2(t), \end{cases}$$

with disturbance bounds $|v_1| \leq \mu_1$, $|v_2| \leq \mu_2$.
For this system

$$\mathscr{B}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathscr{M}_h(t) = \begin{bmatrix} h/2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example

First Approach. A linear transform $\hat{u}_1(t) = hu_1(t)/2 + u_2(t)$, $\hat{u}_2(t) = u_1(t)$:

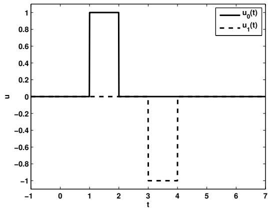
$$\begin{cases} \dot{x}_{h1}(t) = x_{h2}(t) + \hat{u}_1(t) + v_1(t), \\ \dot{x}_{h2}(t) = \hat{u}_2(t) + v_2(t). \end{cases}$$

Control bounds: $|\hat{u}_1| \leq \nu_1$, $|\hat{u}_2| \leq \nu_2$, with $\nu_j \geq \mu_j$.

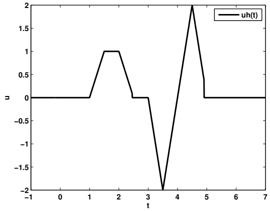
Second Approach.

$$\begin{cases} \dot{x}_1(t) = x_2(t) + u_1(t) + v_1(t), \\ \dot{x}_2(t) = u_2(t) + v_2(t), \end{cases}$$

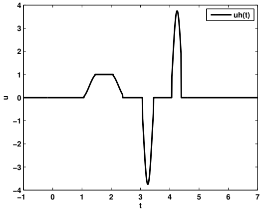
Control bounds: $|u_1| \leq \nu_1$, $|u_2| \leq \nu_2$, with $\nu_j \geq \mu_j$.



realization of control $\mathbf{u}_j(t)$



First Approach



Second Approach

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Thank you for attention!