# The Problem of Output Measurement Feedback Control Under Set-valued Uncertainty : from Theory to Computation

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# OUTLINE

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3. The GCS Problem OF Guaranteed Control Synthesis

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5. Systems with Linear Structure:
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6. Linear Systems : the Solution Scheme,
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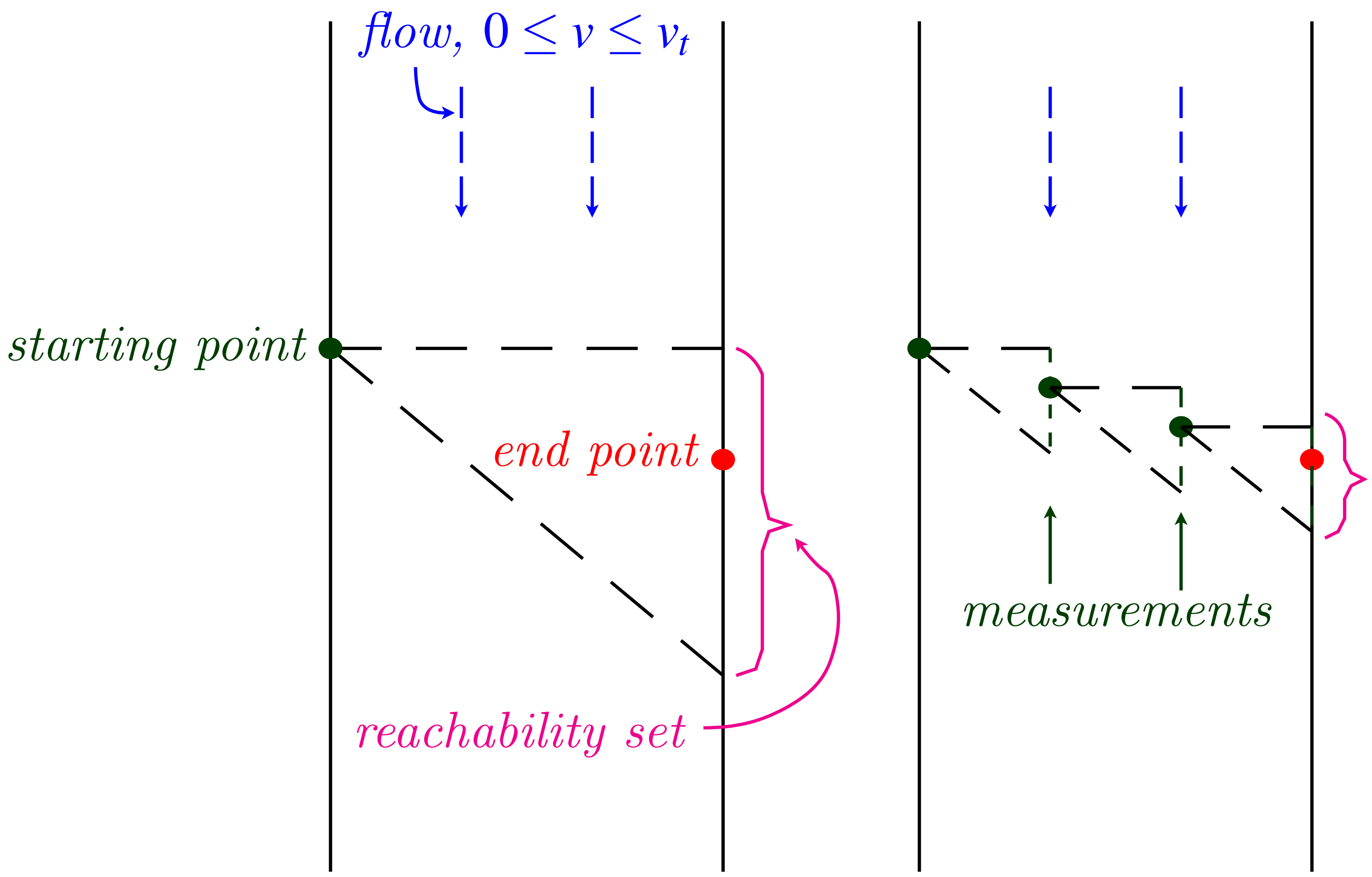
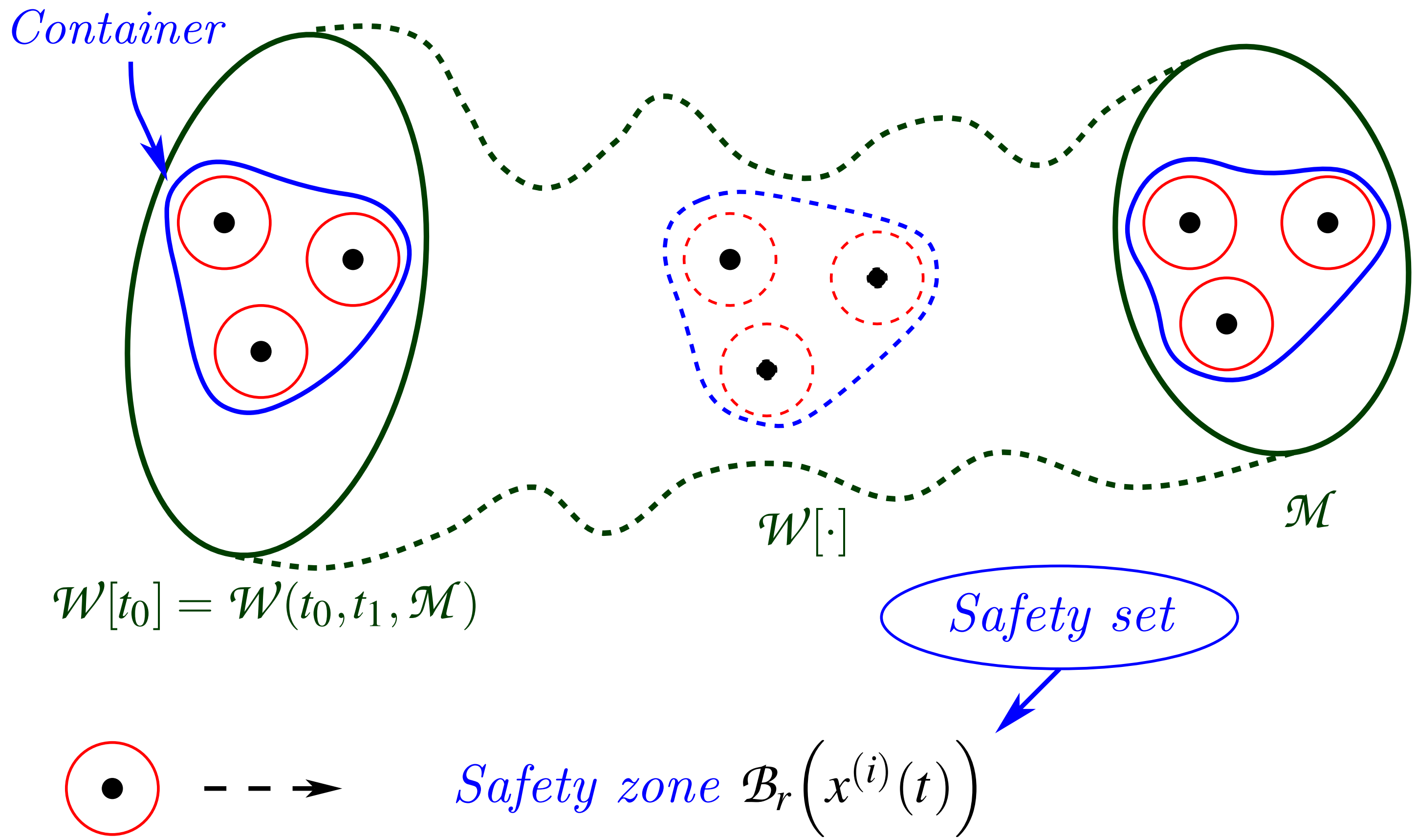
8. Conclusion

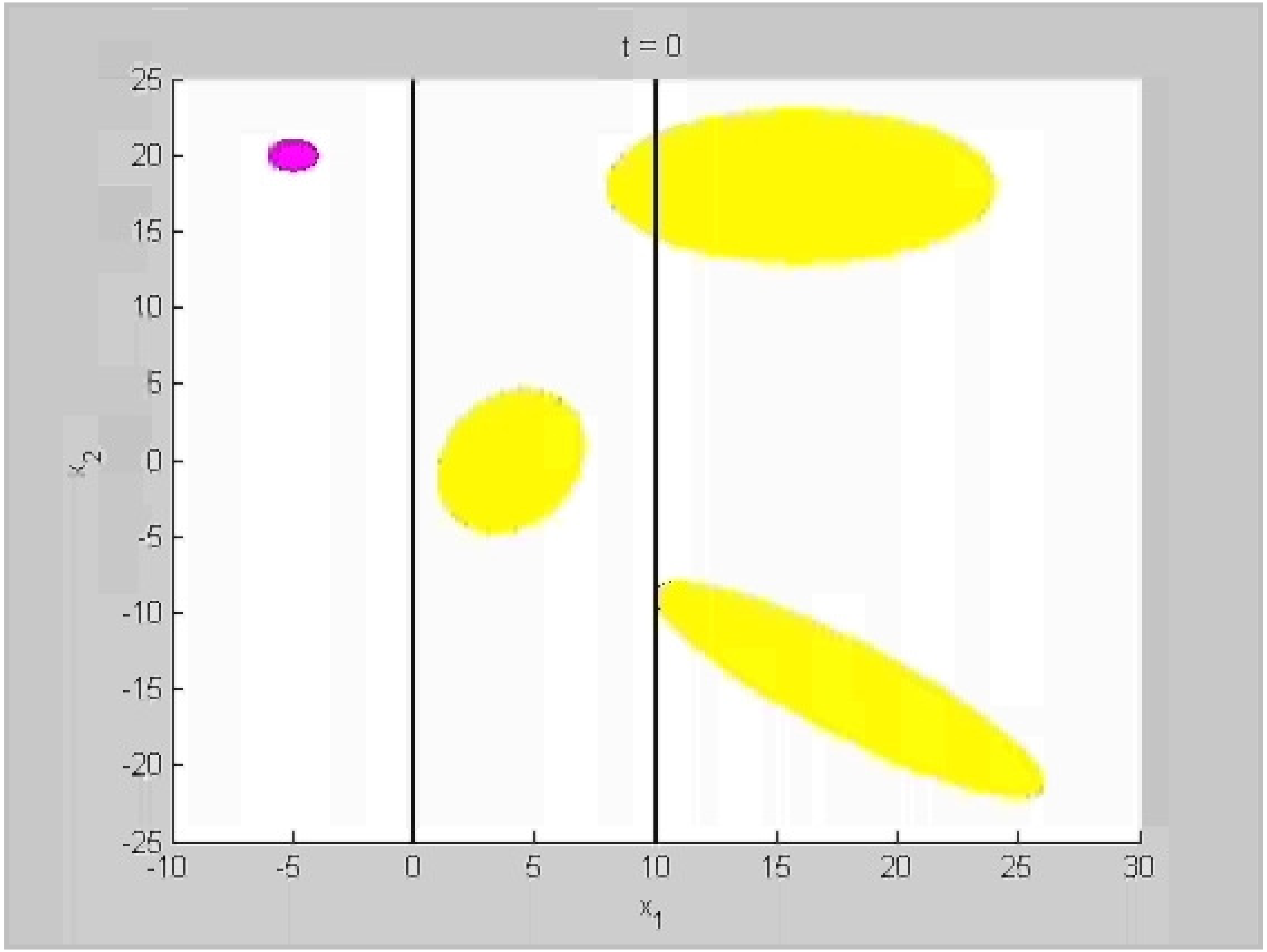
# MOTIVATIONS



flow, $0 \leq v \leq v_t$
starting point
end point
reachability set
measurements

# *Team Control Synthesis*
# *Complete measurements*

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$x_1$

# The System Equations and the Uncertainties

The uncertain system :

$$\frac{dx}{dt} = f_1(t,x,u) + f_2(t,x,v),\ x \in \mathbb{R}^n,\ t \in [t_0, \vartheta] \tag{1}$$

with continuous right-hand sides satisfying conditions of uniqueness an extendibility of solutions.

hard bounds on control $u$ and unknown disturbance $v(t)$:

$$u \in \mathcal{P}(t),\ v(t) \in \mathcal{Q}(t), \tag{2}$$

$\mathcal{P}(t), \mathcal{Q}(t)$ — compact sets in $\mathbb{R}^p, \mathbb{R}^q$,

Hausdorff-continuous.

Measurement equation:

$$y(t) = h(t,x) + \xi(t), \;\; y \in \mathbb{R}^m, \tag{3}$$

measurements — $y(t), \;\; t \in \mathcal{T}$ – (continuous or discrete)

disturbance in measurement $\xi(t)$ — unknown but bounded:

$$\xi(t) \in \mathcal{R}(t), \;\; t \in [t_0, \vartheta], \tag{4}$$

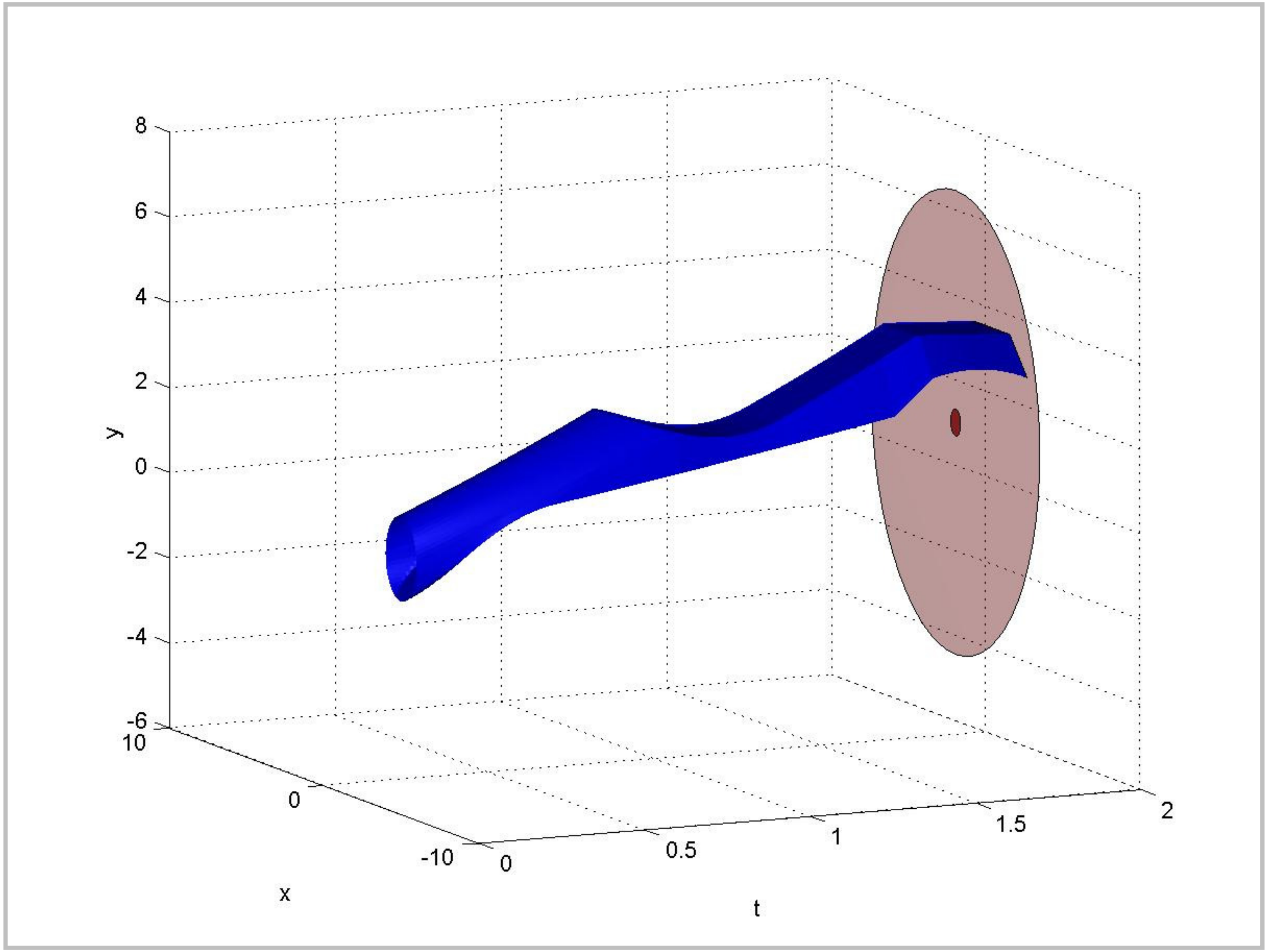
$\mathcal{R}(t)$ — similar to $\mathcal{P}(t), \;\; h(t,x)$ — continuous.

Initial condition:

$$x(t_0) \in X^0, \tag{5}$$

$X^0$ — compact.

Starting Position: $\{t_0, X^0\}$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# BASIC PROBLEM

## STEER SYSTEM

$$\frac{dx}{dt} = f_1(t,x,u) + f_2(t,x,v),\ x \in \mathbb{R}^n,\ t \in [t_0, \vartheta], \qquad (1)$$

$$y(t) = h(t,x) + \xi(t),\ ;\ y \in \mathbb{R}^m, \qquad (3)$$

from starting position $\{t_0, \mathcal{X}^0\}$ to terminal position $\{\vartheta, \mathcal{M}\}$,

by **feedback control strategy** $\mathcal{U}(t,\cdot)$,

on the basis of available information:

- system model : equations (1), (3),
- starting position $\{t_0, \mathcal{X}^0\}$,
- available measurement $y(t)$,
- given **constraints** on **control** $u$ and

uncertain **disturbance inputs** $v(t), \xi(t)$

What should the NEW STATE of the SYSTEM be ?

*** Classical case under complete information:

Position (state) – $\{\mathbf{t}, \mathbf{x}\}$ – single valued

Closed-loop control : $\{\mathbf{u}(\mathbf{t}, \mathbf{x})\}$

Trajectories – single-valued : $x[t] = x(t, t_0, x^0)$.

*** Output feedback control under incomplete information:

with – set-valued bounds (no statistical data available):

Position (state) – set-valued: $\mathcal{X}[t]$

On-line set-valued position (NEW STATE) of the system may be taken as:

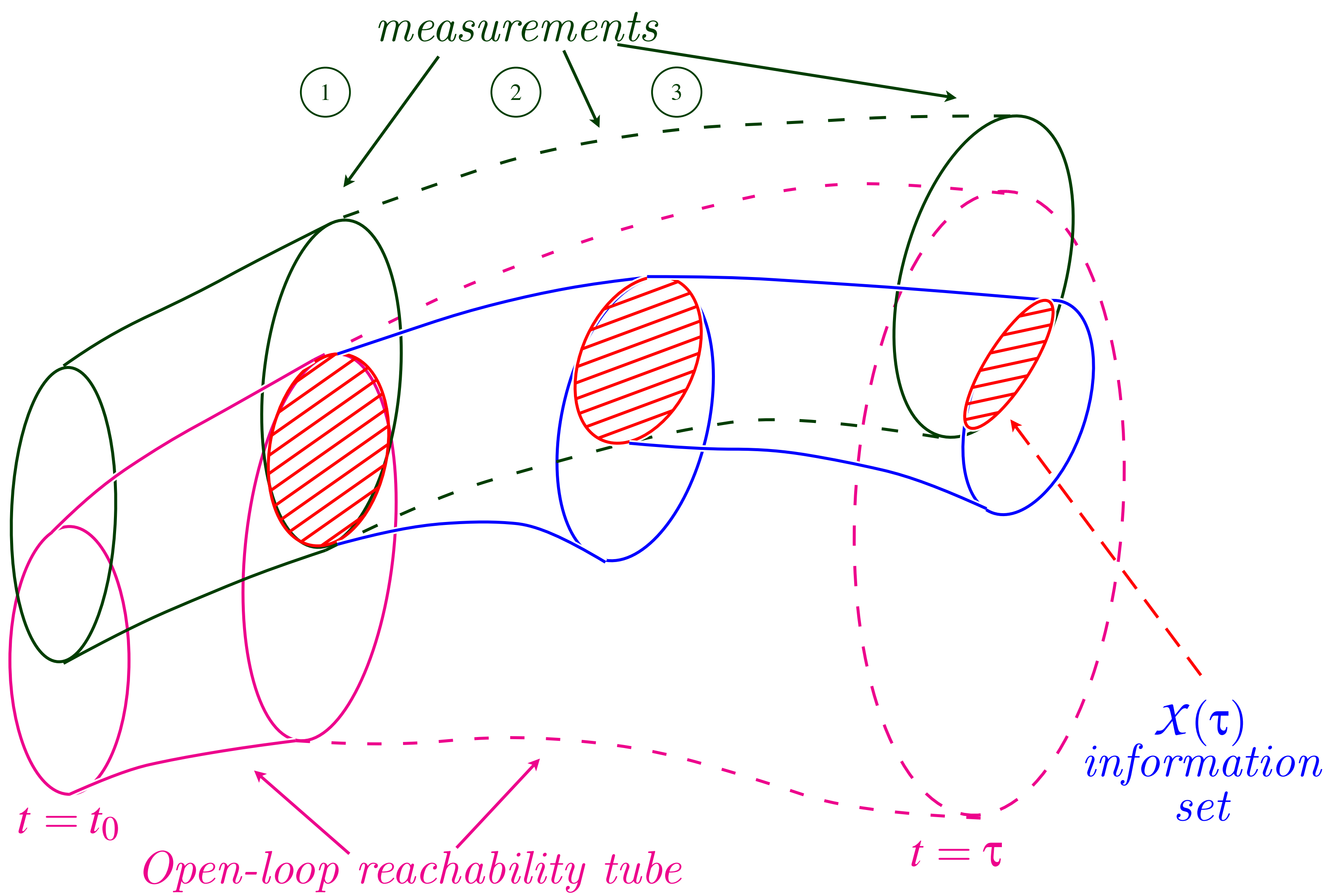
* $\{t, y_t(\cdot)\}$ — memorize measurements,

(in stochastic control this is done through observers and filters (Kalman))

** $\{t, \mathcal{X}[t]\}$ — find set-valued information set consistent with measurements and constraints on uncertain items: find set-valued information tubes

*** $\{t, V(t, \cdot)\}$ — find information state – function $V(t,x)$ such that $\mathcal{X}[t] = \{x : V(t,x) \leq \alpha\}$ is the level set of $V(t,x)$, (found through Hamilton-Jacobi-Bellman (HJB) PDE equations).

# *Guaranteed State Estimation under Set-membership noise*



**Problem I** of Measurement Output Feedback Control:

Specify feedback strategy (closed-loop controls) $U(t,\mathcal{X}[t])$ or $U(t,V(t,\cdot))$

which steers overall system

FROM any starting position $\{\tau,\mathcal{X}[\tau]\}$, $\tau \in [t_0,\vartheta]$

TO given neighborhood $\mathcal{M}_\mu$ of target set $\mathcal{M}$ at time $\vartheta$ :

$$\{\tau,\mathcal{X}[\tau]\} \to \{\vartheta,\mathcal{X}[\tau]\},\ \ \mathcal{X}[\vartheta] \subseteq \mathcal{M}_\mu$$

despite unknown disturbances and incomplete measurements.

**ATTENTION for MATHEMATICIANS:** $\mathcal{U} = \{U(t,\mathcal{X}[t])\}$ must ensure the existence and extendability of solutions to differential inclusion

$$\dot{x} \in f_1(t,x,U(t,\mathcal{X}[t])) + f_2(t,x,v),$$

within interval $t \in [t_0,\vartheta]$, whatever be $v(t)$.

# (Measurement) Output Feedback Control

Closed-loop (feedback) control strategies: $\mathcal{U}(t,\mathcal{X}),\ \ \mathcal{U}(t,V(t,\cdot))$,

with state $\{t,\mathcal{X}\}$, or $\{t,V(t,\cdot)\}$,

and trajectories –set-valued: $\mathcal{X}[t]=\mathcal{X}(t,t_0,\mathcal{X}^0)$

or single valued $x[t]$, with set-valued error-bound $\mathcal{R}[t]$,

with state $\{t,x[t],\Omega[t]\}$ (external estimate $\mathcal{E}[t]\supseteq\mathcal{R}[t]$).

trajectories $x[t]=x(t,t_0,x^0)$,

error bounds $\Omega[t]=\Omega(t,t_0,\mathcal{X}^0-x^0)$.

**REMARK:** Problem I may be separated into:

Problem GSE of guaranteed state estimation(finite-dimensional)

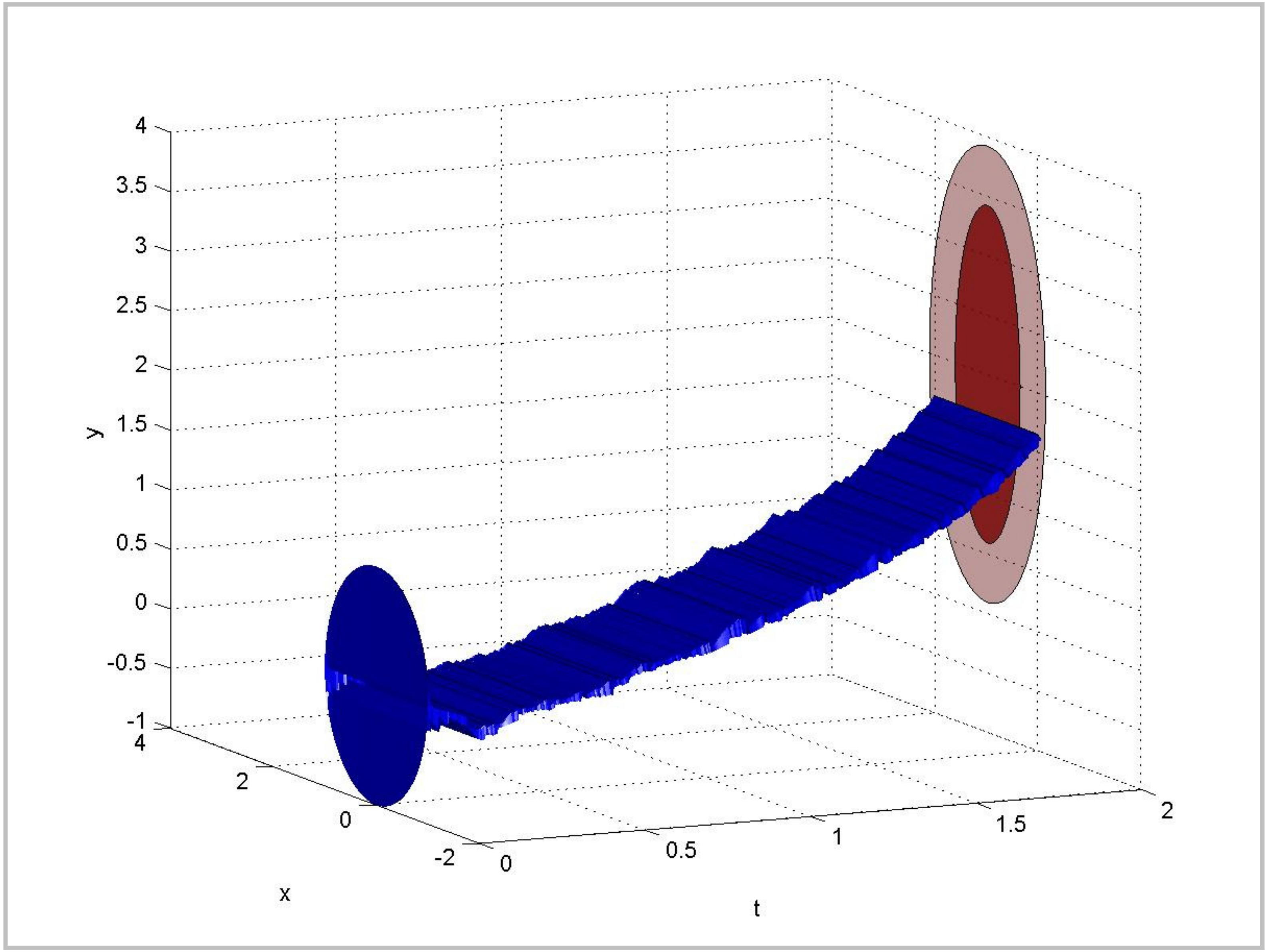
and

Problem GCS of guaranteed control synthesis (infinite-dimensional)

**OUR AIM** :

(a) Find possibility of solutions while avoiding infinite-dimensional schemes.

(b) Design feasible computational methods.



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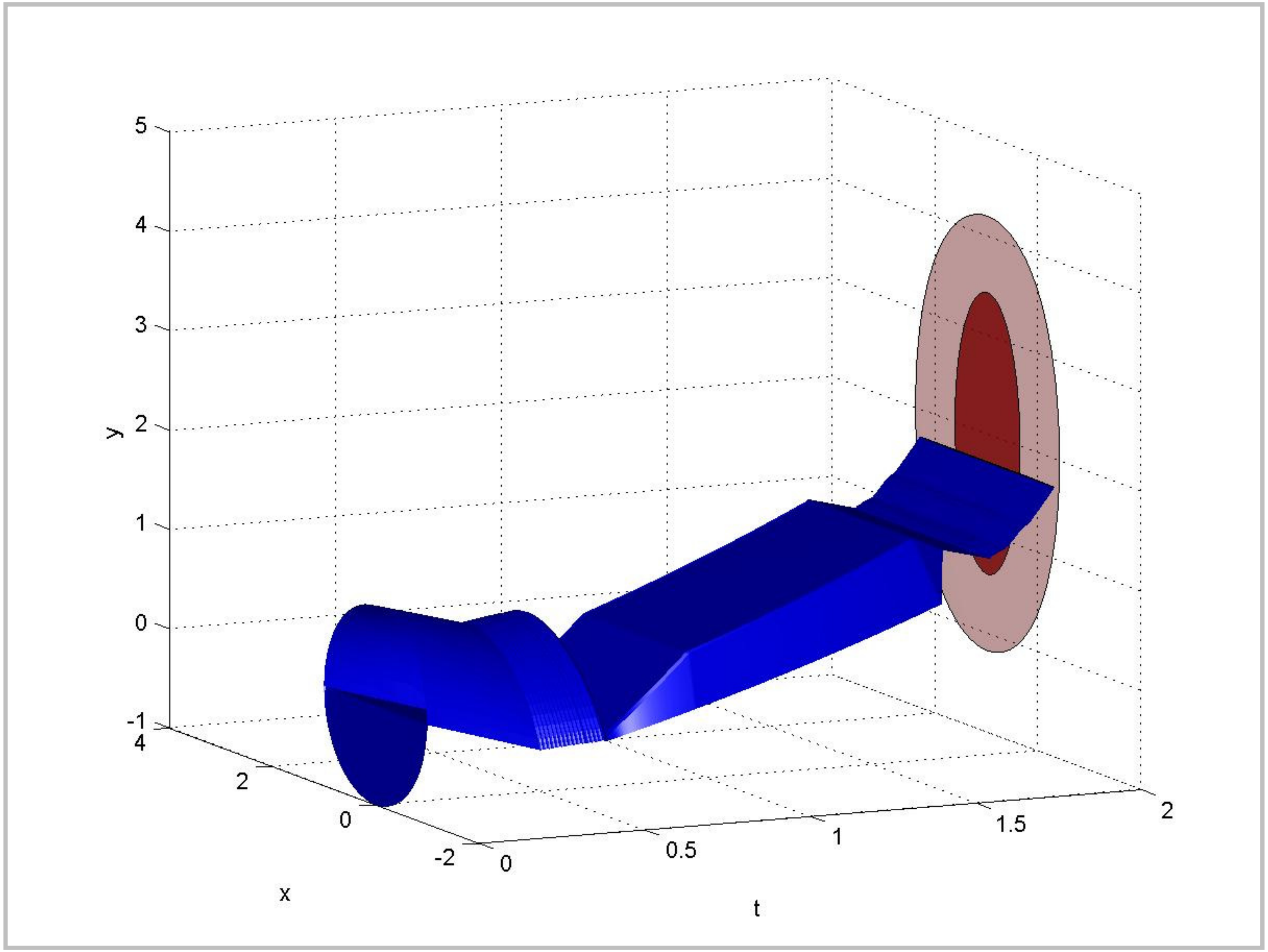
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# SOLUTION METHODS

(a) GENERAL METHOD:

the HAMILTON-JACOBI-BELLMAN (HJB) EQUATIONS

(b) USING INVARIANT SETS and AIMING METHODS

SET-VALUED CALCULUS+ NONLINEAR ANALYSIS

FOR LINEAR SYSTEMS: CONVEX ANALYSIS

(c) THE $H$-INFINITY APPROACH

(d) APPROXIMATE METHODS:

THE COMPARISON PRINCIPLE, DISCRETIZATION METHODS

(e) COMPUTATION METHODS FOR LINEAR SYSTEMS:

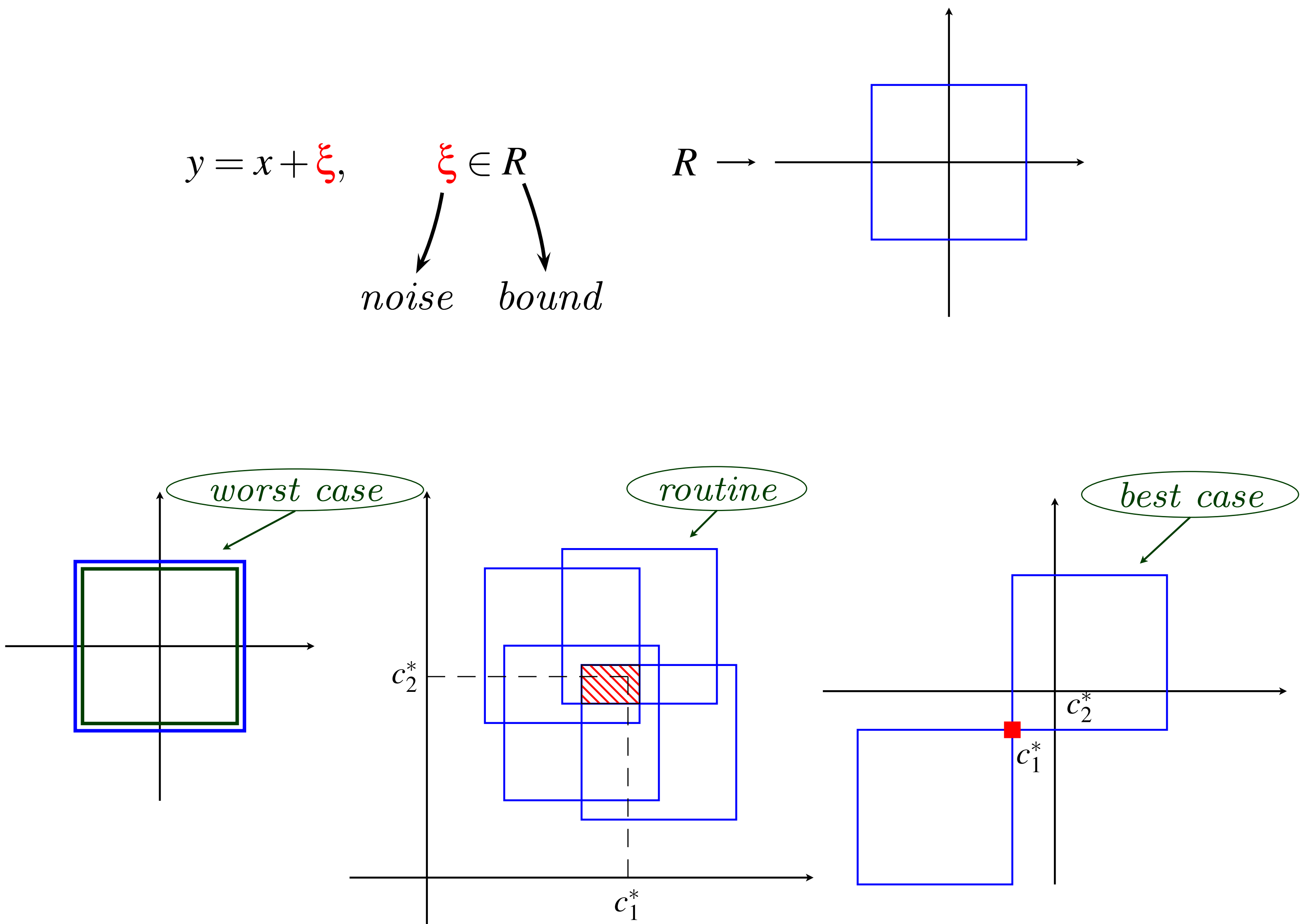
ELLIPSOIDAL CALCULUS, POLYHEDRAL CALCULUS or BOTH

(f) INTERTWINING THE ABOVE METHODS

## Problem GSE of Guaranteed State Estimation

## The One-Stage Problem

NOTE THAT THERE IS WORST CASE NOISE and BEST CASE NOISE


$y = x + \xi$,
$\xi \in R$
noise
bound
$R \rightarrow$
worst case
routine
best case
$c_2^*$
$c_1^*$
$c_2^*$
$c_1^*$

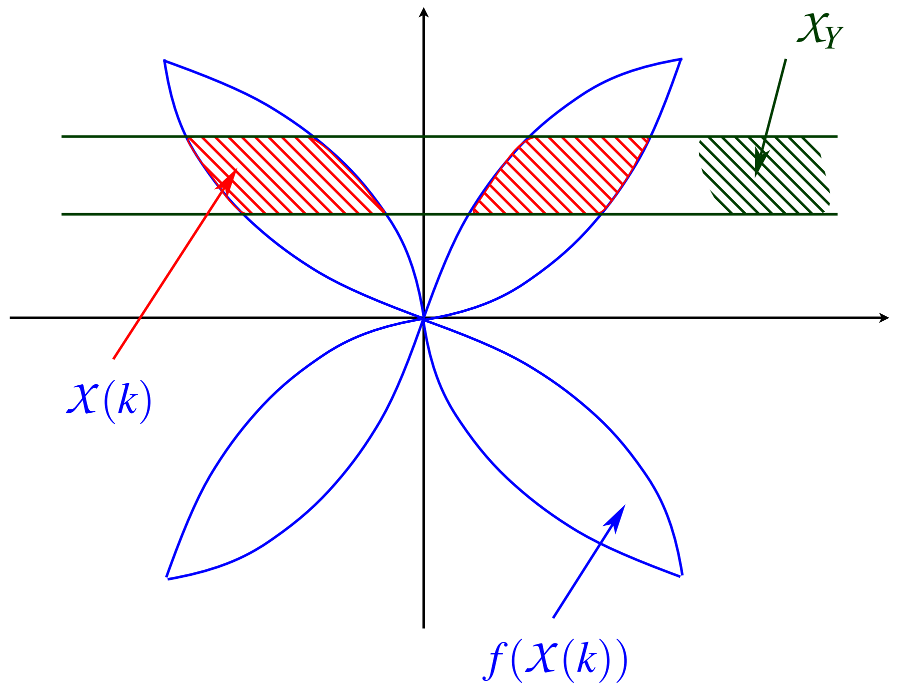
# *Examples: nonlinear maps*

$$\begin{cases} x(k+1) &= \underline{f}(x(k)) \\ \underline{y(k+1)} &= \underline{G}x(k+1)+\xi \end{cases}$$

$$\textit{Take } x \in \mathbb{R}^2, \quad f(x) = \begin{pmatrix} ax_1 & x_2^2 \\ ax_1^2 & x_2 \end{pmatrix},$$

$$\mathcal{X}(k) = \{x \in \mathbb{R}^2 \ : \ |x_i| \leq 1; i = 1,2\}, \quad y(k+1) = x_2(k)+\xi, \quad |\xi| \leq \mu,$$

$$\mathcal{X}_Y(k+1) = \{x \ : \ x \in [y(k+1)+\mu, y(k+1)-\mu]\}, \quad \mathcal{X}(k+1) = f(\mathcal{X}(k)) \cap \mathcal{X}_Y(k+1)$$



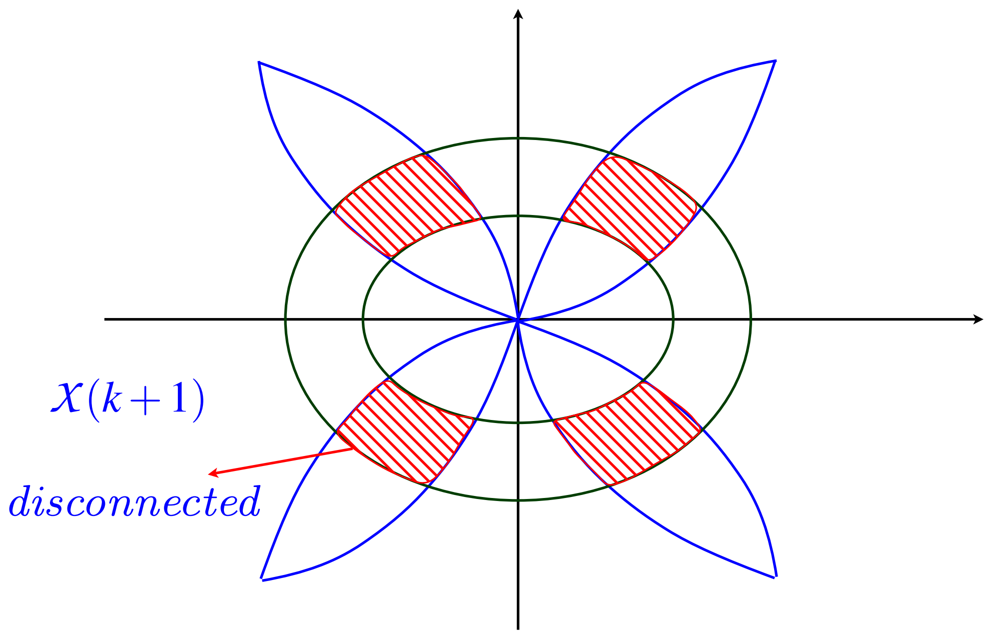
# Nonlinear Examples

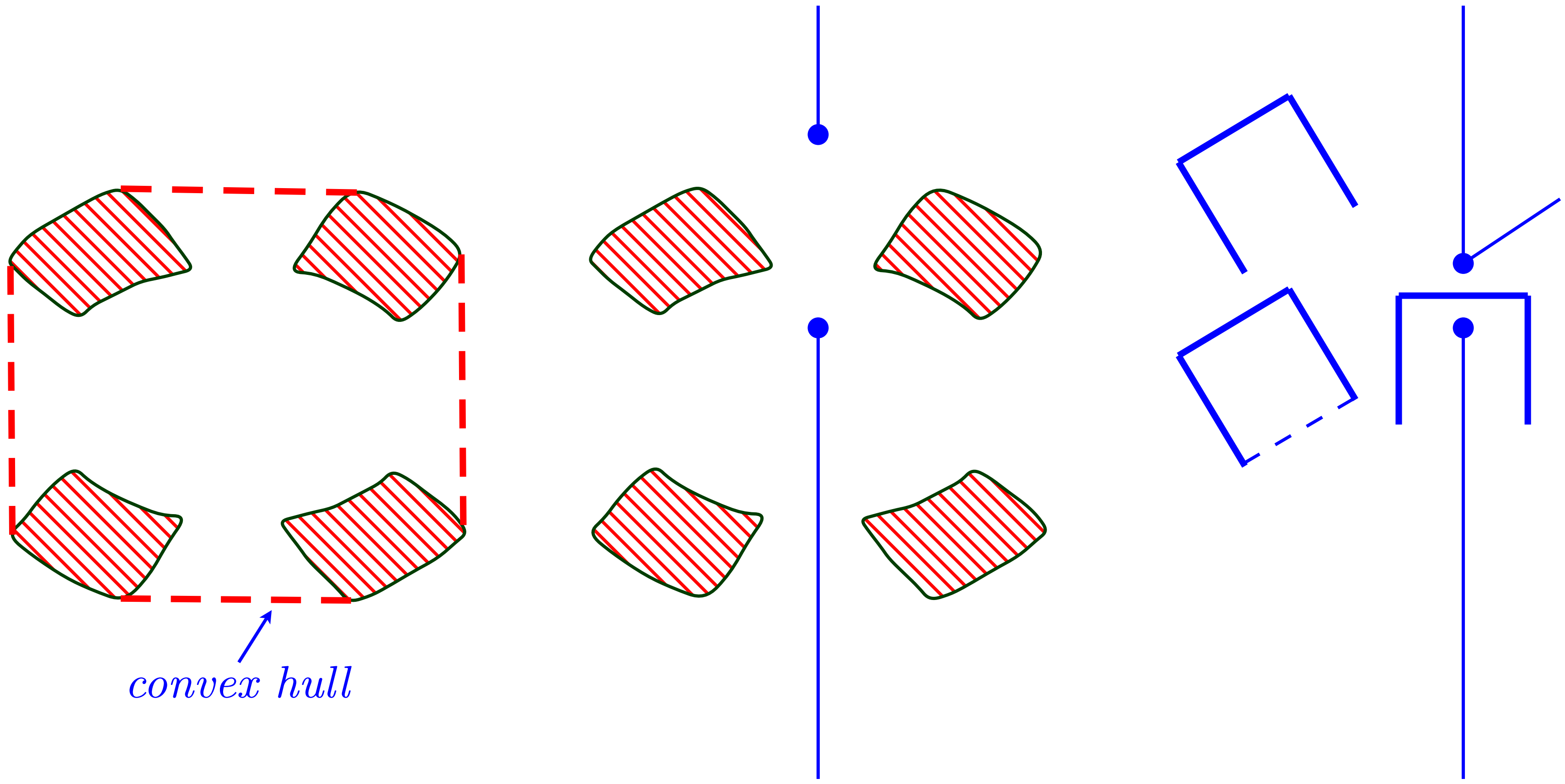
$$\begin{cases} x(k+1) = f(x(k)) \\ y(k+1) = a^2x_1^2 + b^2x_2^2 + \xi \end{cases} \qquad |\xi| \leq \varepsilon$$

$$\mathcal{X}(k) = \{x \in \mathbb{R}^2 \,:\, |x_i| \leq 1; i = 1,2\}$$

$$\mathcal{X}_Y(k+1) = \{x \,:\, x \in [y(k+1) - \varepsilon, y(k+1) + \varepsilon]\}$$

$$\mathcal{X}(k+1) = f(\mathcal{X}(k)) \cap \mathcal{X}_Y(k+1)$$





convex hull

# Unkown but bounded noise

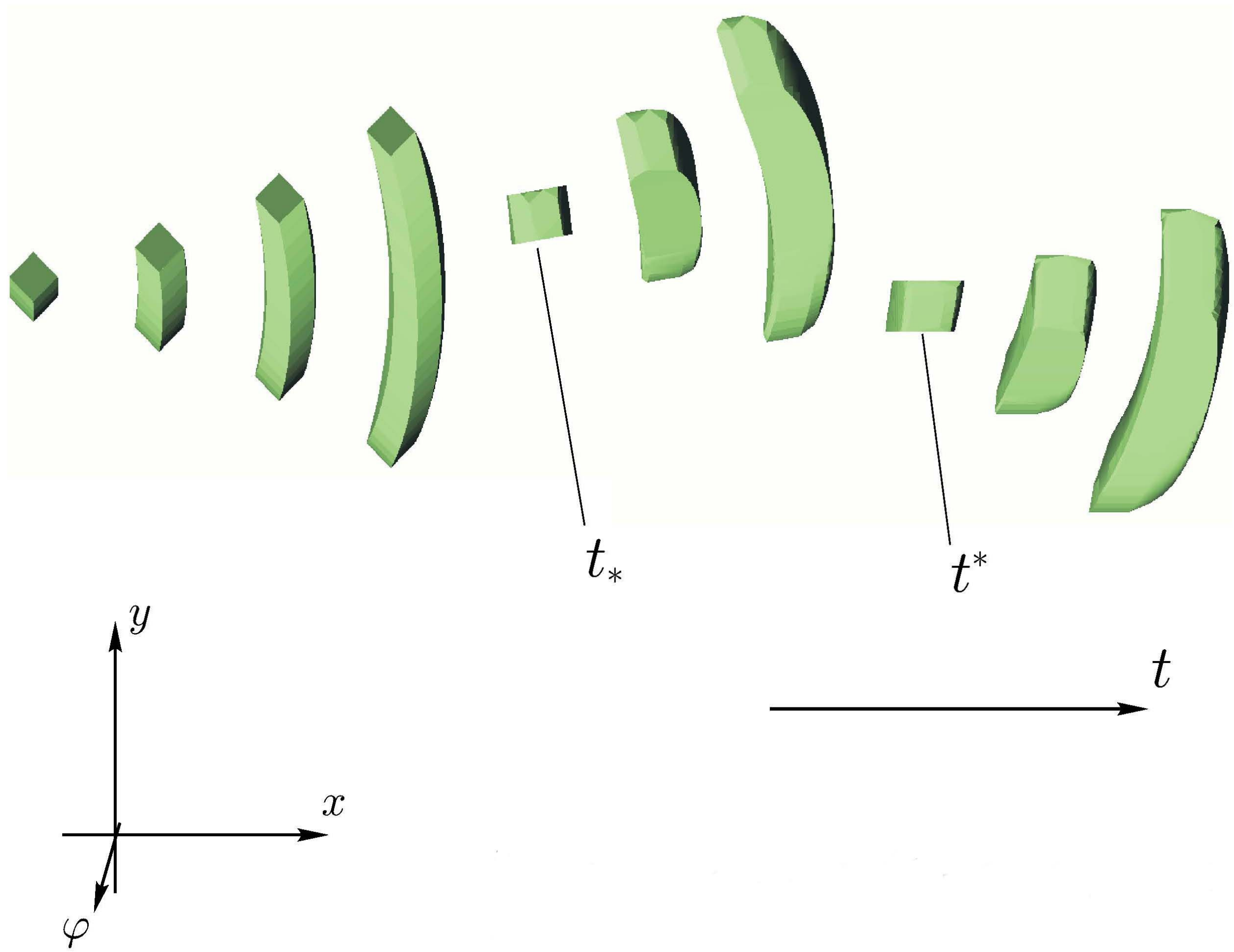
(i)Measurements – at given time (continuous or discrete). Noise – unknown, with given bounds.

Has a **worst case** when $\mathcal{W}[t]$ is **largest possible** and a best case when $\mathcal{W}[t]$ may even reduce to a point

(ii) Measurements arrive at random instants of time, due to distribution of Poisson. Noise - with given bounds and given probabilistic density.

With stochastic noise the worst and best cases arrive with probability zero. The statistical estimates of $x$ are consistent.

# The Dynamics of the Information Set




$t_*$ and $t^*$ are the instants of discrete observations

# Problem GSE of Guaranteed (“Minmax”) State Estimation

Problem GSE may be formulated in two versions - $E_1$ and $E_2$

**Problem $E_1$:** Given are equations

$$\frac{dx}{dt} = f_1(t,x,u) + f_2(t,x,v), \quad y(t) = h(t,x) + \xi(t) \qquad (i)$$

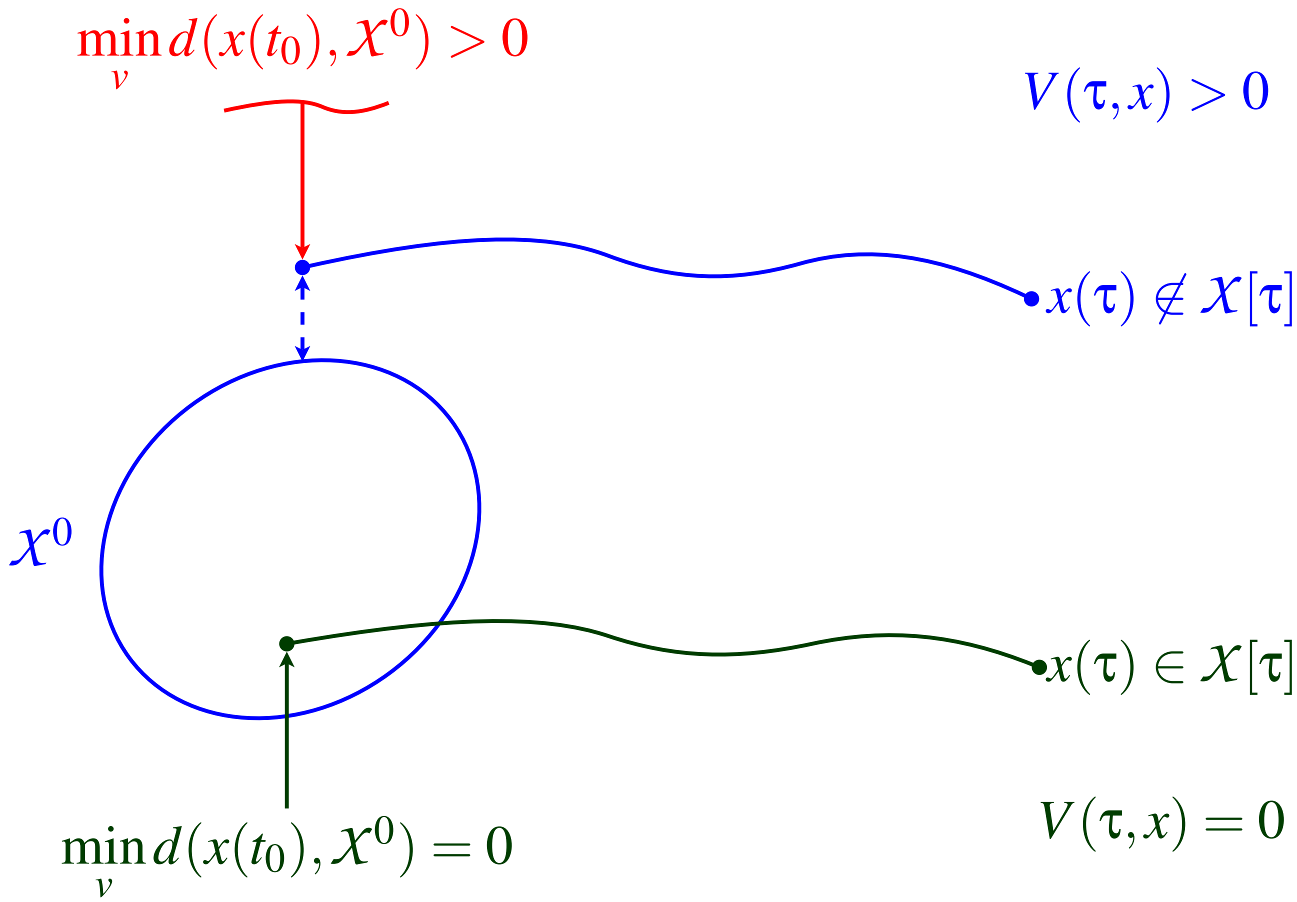
position $\{t_0, \mathcal{X}^0\}$, used control $u[s], s \in [t_0, \tau)$, measurement $y = y^*(t)$, $t \in [t_0, \tau]$, and constraints

$$u \in \mathcal{P}, \; v \in \mathcal{Q}, \; \xi \in \mathcal{R} \qquad (ii)$$

with $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ given.

Specify information set $\mathcal{X}[\tau]$, of solutions $x(\tau)$ to system (i), consistent with system equations, measurement $y^*(t)$, $t \in [t_0, \tau]$ and constraints (ii).

The information set $\mathcal{X}[\tau]$ is the guaranteed estimate of $x(\tau)$.




$$V(\tau, x) = \min_v d(x(t_0), \mathcal{X}^0)$$

It is necessary not only **to calculate set** $\mathcal{X}[\tau]$, but to arrange

on-line calculations , following the evolution of $\mathcal{X}[t]$ in time.!!!

This leads to the problem of DYNAMIC OPTIMIZATION:

**Problem** $E_2$ Given starting position $\{t_0, \mathcal{X}^0\}$, and realization $y^*(s),\ s \in [t_0, \tau]$,

Find value function:

$$V(\tau, x) = \min_{v}\{d(x(t_0), \mathcal{X}^0) \,|v(t) \in \mathcal{Q}(t), t \in [t_0, \tau]\}$$

due to equation (1), under additional conditions

$$x(\tau) = x;\ \ y^*(s) - h(s, x(s)) \in \mathcal{R}(s),\ s \in [t_0, \tau].$$

The last condition is actually an on-line state constraint

The following relation is true

$$\mathcal{X}[t] = \{x : V(t,x) \leq 0\} \quad !!!$$

The value function $V(t,x)$ may be found by solving an HJB equation!

Introduce notation $V(\tau,x) = V(\tau,x|V(t_0,\cdot))$,

Then the principle of optimality for problem GSE reads:

$$V(\tau,x|V(t_0,\cdot)) = V(\tau,x|V(t,\cdot|V(t_0,\cdot))),\ t_0 \leq t \leq \tau. \quad (!)$$

This allows to derive an HJB (Dynamic Programming) equation, to calculate $V(t,x)$.

The HJB equation:

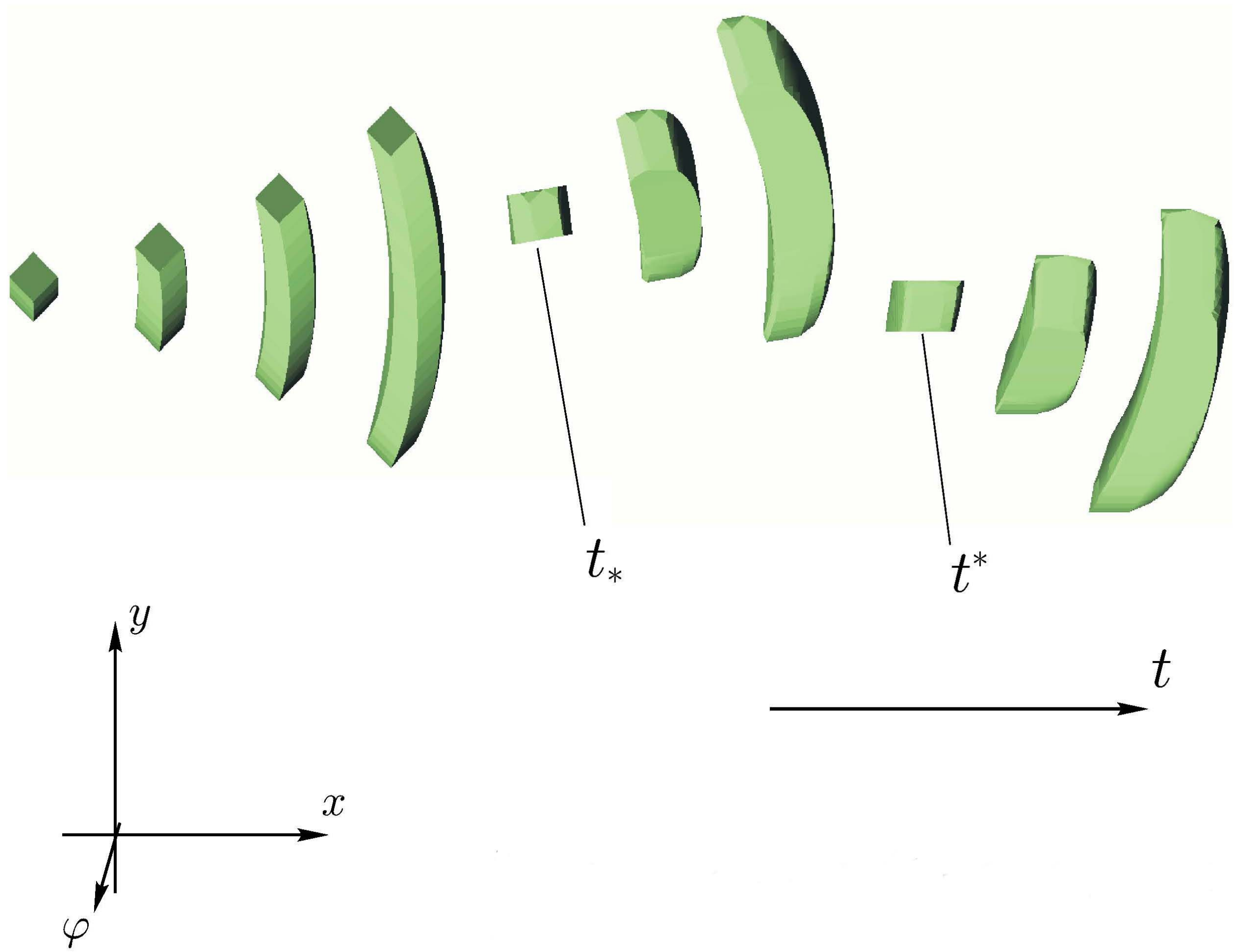
$$\frac{\partial V}{\partial t} + \max_{v} \left\{ \left( \frac{\partial V}{\partial x}, f_1(t,x,u^*(t)) + f_2(t,x,v) \right) - \right.$$

$$\left. -d^2(y^*(t) - h(t,x), \mathcal{R}(t)) \,\middle|\, v(t) \in Q(t) \right\} = 0,$$

under boundary condition $V(t_0,x) = d^2(x, \mathcal{X}^0)$.

Discretized scheme: $\mathcal{X}[t+\sigma] \sim \mathcal{X}[t+\sigma-0] \cap Y(t+\sigma)$

# The Dynamics of the Information Set




$t_*$ and $t^*$ are the instants of discrete observations

## Problem GCS of Guaranteed Synthesizing Control

The “motion” of the evolving system is given by either

the tube $\mathcal{X}[\tau]$ or the function $V(\tau,\cdot)$.

**Problem** *GCS*. Find value function

$$\mathcal{V}(\tau, V(\tau,\cdot)) = \min_u \max_y \left\{ d^2(x[\vartheta], \mathcal{M}) \,\middle|\, u \in \mathcal{U},\ y(\cdot) \in Y(\cdot,u) \right\}$$

over closed-loop controls and all predicable ”future” tubes
$Y(\cdot,u) = Y(\vartheta,\tau;\mathcal{X}[\tau],u)$.

Value function $\mathcal{V}(\tau,x) = \mathcal{V}(\tau,V(\tau,\cdot))$ satisfies the (infinite-dimensional)

Principle of Optimality in metric space of functions $V(\cdot)$:

$$\mathcal{V}(\tau,V(\tau,\cdot)) = \mathcal{V}(\tau,V(\tau,\cdot)|\vartheta,\mathcal{V}(\vartheta,\cdot))$$

Finding $\mathcal{V}(t,V(t,\cdot))$ produces the solution strategy

$$u = u^0(t,V(t,\cdot)) \in \mathcal{U}$$

But to find $\mathcal{V}(\tau,V(\tau,\cdot))$ one would have to solve a PDE in the space of functions rather than in finite dimensions.

The solution strategy

$$u = u^0(t, V(t,\cdot)) \in \mathcal{U}$$

guarantees condition

$$\mathcal{V}(\tau, V(\tau,\cdot)) \leq \max_y \max_x \{d^2(x, \mathcal{M}) \mid V(\vartheta, x | V(\tau,\cdot)) \leq 0\} \mid u \in \mathcal{U};\, y(\cdot) \in Y(\tau, u)\}$$

for **any** strategy $u = u(t, V(t,\cdot)) \in \mathcal{U}$.

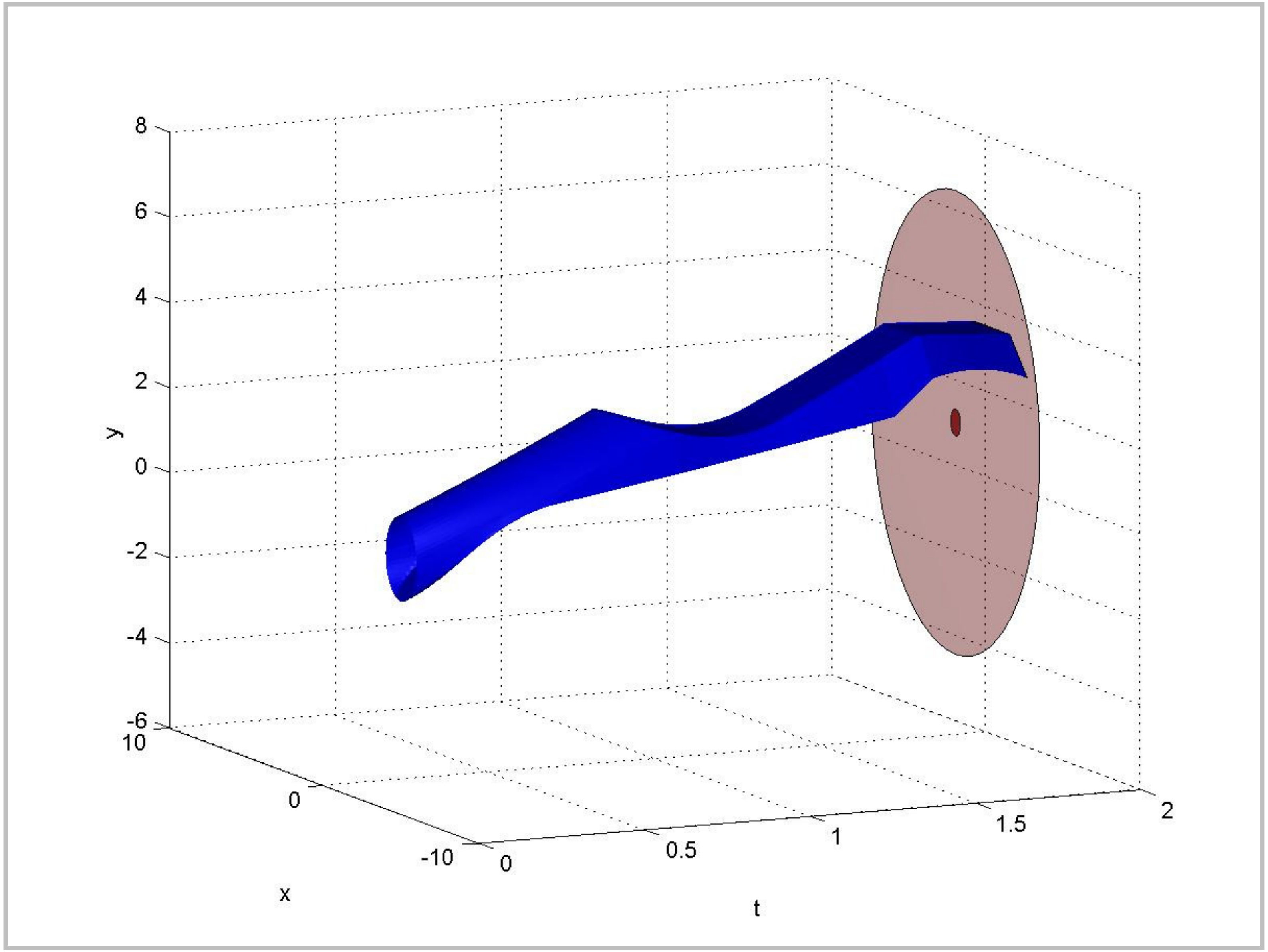
Note that $V(t,\cdot)$ are the ”motions” of the formal evolution of $\mathcal{X}[t] = \{x : V(t,x) \leq 0\}$ – the on-line STATE of the system.

A straightforward application of the DP approach may demand a heavy computational burden, however there are promising approaches, such as level set methods, the comparison principle, **discretization techniques** and others

BUT DO NOT HURRY TO DISCARD DP:(!!!)

IN CASE of LINEAR SYSTEMS and
QUITE A NUMBER OF NONLINEAR
THE EXACT SOLUTION MAY BE REACHED
WITHOUT INFINITE–DIMENSIONAL PDE's,
BUT ONLY THROUGH FINITE-DIMENSIONAL SCHEMES(!!!)

The computations are of course all designed within finite-dimensional schemes.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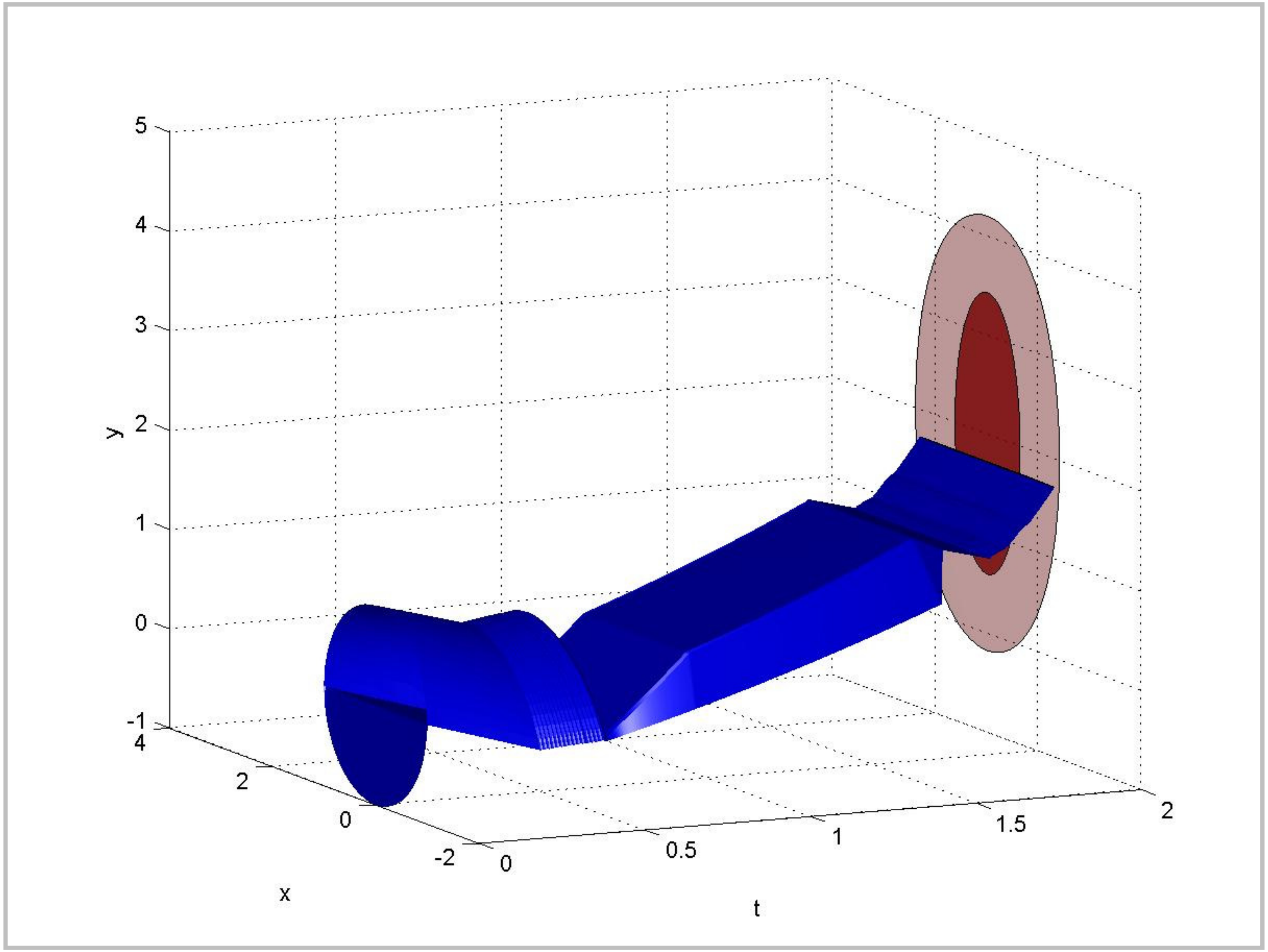
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# II. Linear Systems under Hard Bounds

The uncertain linear system :

$$dx/dt = A(t)x + B(t)u + C(t)v(t), (L1)$$

with continuous matrix coefficients $A(t), B(t), C(t)$

hard bounds on control $u$ and disturbance $v(t)$:

$$u \in \mathcal{P}(t),\ v(t) \in \mathcal{Q}(t),\ t \in [t_0, \vartheta]$$

$\mathcal{P}(t), \mathcal{Q}(t)$ — convex compact sets in $\mathbb{R}^p, \mathbb{R}^q$,

Hausdorff-continuous.

Measurement equation:

$$y(t) = H(t)x + \xi(t), \;\; \mathrm{rank} H = m, (L2)$$

disturbance $\xi(t)$ — unknown but bounded:

$$\xi(t) \in \mathcal{R}(t), \;\; t \in [t_0, \vartheta],$$

$\mathcal{R}(t)$ – convex, compact, Hausdorff-continuous; $H(t)$ — continuous.

Initial condition:

$$x(t_0) \in \mathcal{X}^0,$$

$\mathcal{X}^0$ — convex compact.

Starting Position: $\{t_0, \mathcal{X}^0\}$

## Problem GCS of Output Feedback Control:

Specify feedback control strategy $U(t, \mathcal{X}[t])$ or $U(t, V(t, \cdot))$

which steers overall system

FROM any starting position $\{\tau, \mathcal{X}[\tau]\}$, $\tau \in [t_0, \vartheta]$

TO given neighborhood $\mathcal{M}_\mu$ of target set $\mathcal{M}$ at time $\vartheta$ :

$$\{\tau, \mathcal{X}[\tau]\} \to \{\vartheta, \mathcal{X}[\vartheta]\}, \;\; \mathcal{X}[\vartheta] \subseteq \mathcal{M}_\mu$$

despite unknown disturbances and incomplete measurements.

**NOTE:** $\mathcal{U} = \{U(t, \mathcal{X}[t])\}$ must ensure existence and extendability of solutions to differential inclusion

$$\dot{x} \in A(t)x + B(t)U(t, \mathcal{X}[t]) + C(t)v(t),$$

within interval $t \in [t_0, \vartheta]$, whatever be $v(t)$.

**New coordinates to simplify calculations**:

– Take transformation $x = G(t,\vartheta)\mathbf{x}$ where $G(t,\vartheta)$ is the fundamental transition matrix for the original homogeneous system (1),

– make necessary changes, then return to original notations.

Then

$$\dot{x} = B(t)u + C(t)v(t),$$

$$y(t) = H(t)x + \xi(t),$$

$$x(t_0) \in X^0 = \mathcal{X}^0$$

under hard bounds of type (2), (4), (5).

Rearrange last system as follows:

$$dx^*/dt = B(t)u,\ x^*(t_0) = 0, \qquad (a)$$

$$d\omega/dt = C(t)v(t),\ v(t) \in Q(t),\ \omega(t_0) \in \mathcal{X}^0, \qquad (b)$$

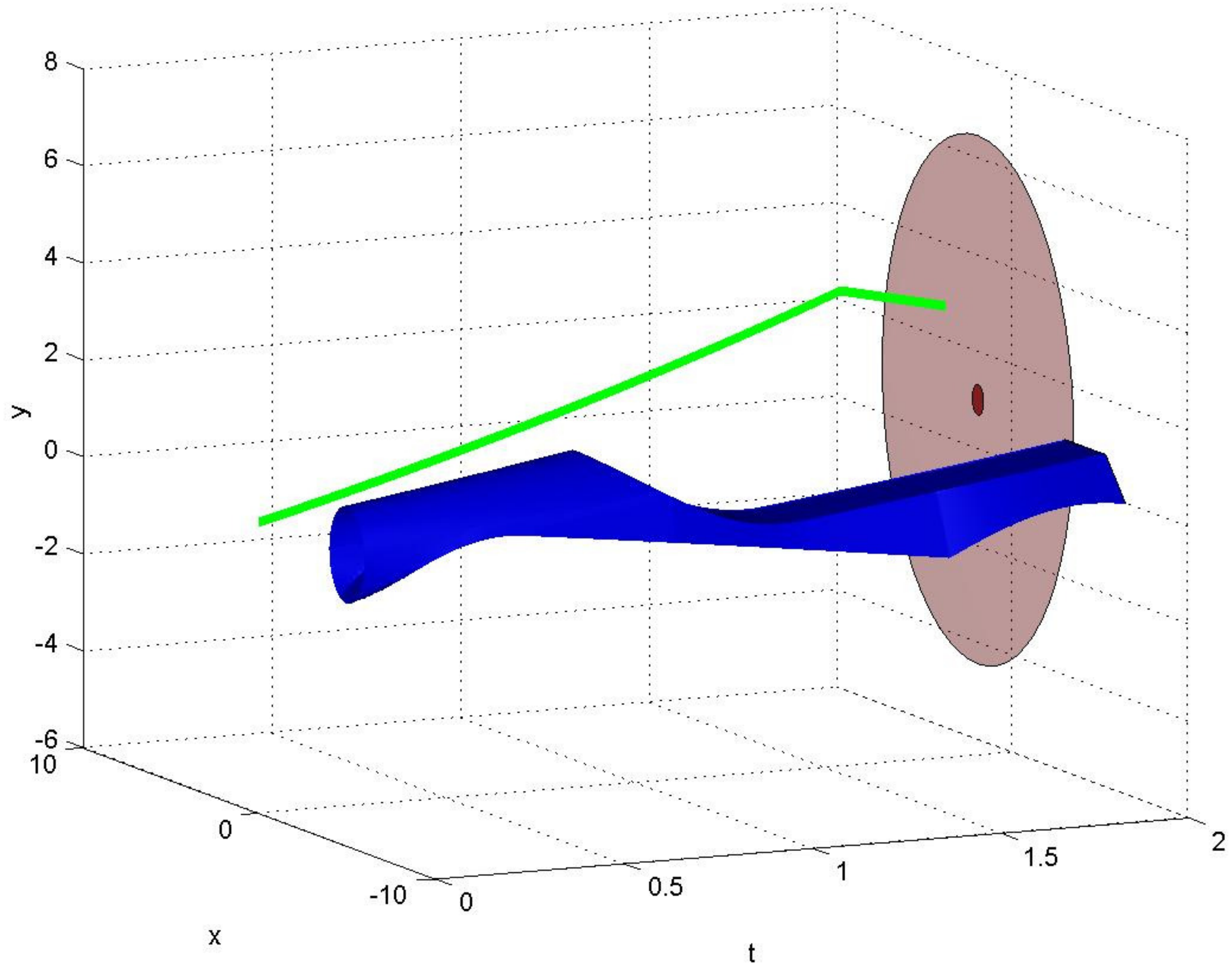
$$z(t) = H(t)\omega + \xi(t),\ \xi(t) \in \mathcal{R}(t) \qquad (c)$$

$$x^* + \omega = x,\ \ z(t) = y(t) - H(t)\int_{t_0}^{t} B(s)u(s)ds = z(t).$$

With $u = u^*(s),\ s \in [t_0, t)$ given, there is a one-to-one mapping between $y(s)$ and $z(s)$.

Define information set for system (a)-(c): $\Omega(t,\cdot) = \Omega[t]$. Then

$$\mathcal{X}[t] = x^*(t) + \Omega[t]$$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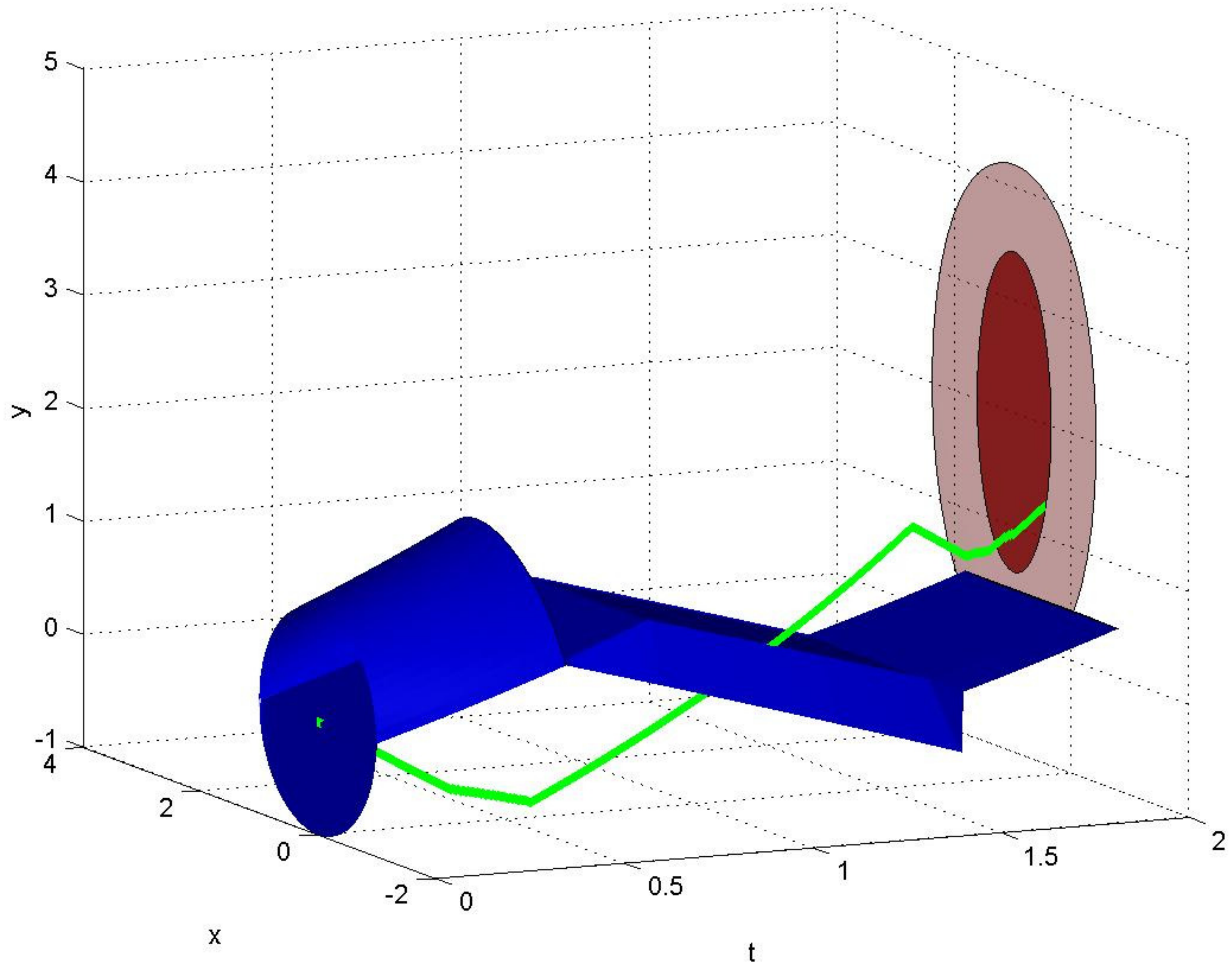

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SOLUTION METHOD:

combining

HJB-techniques with calculating weakly invariant sets

The convex information sets $\mathcal{X}$ and information states $V(t,x)$ (convex functions) may be calculated through HJB equations or convex analysis or their approximations.

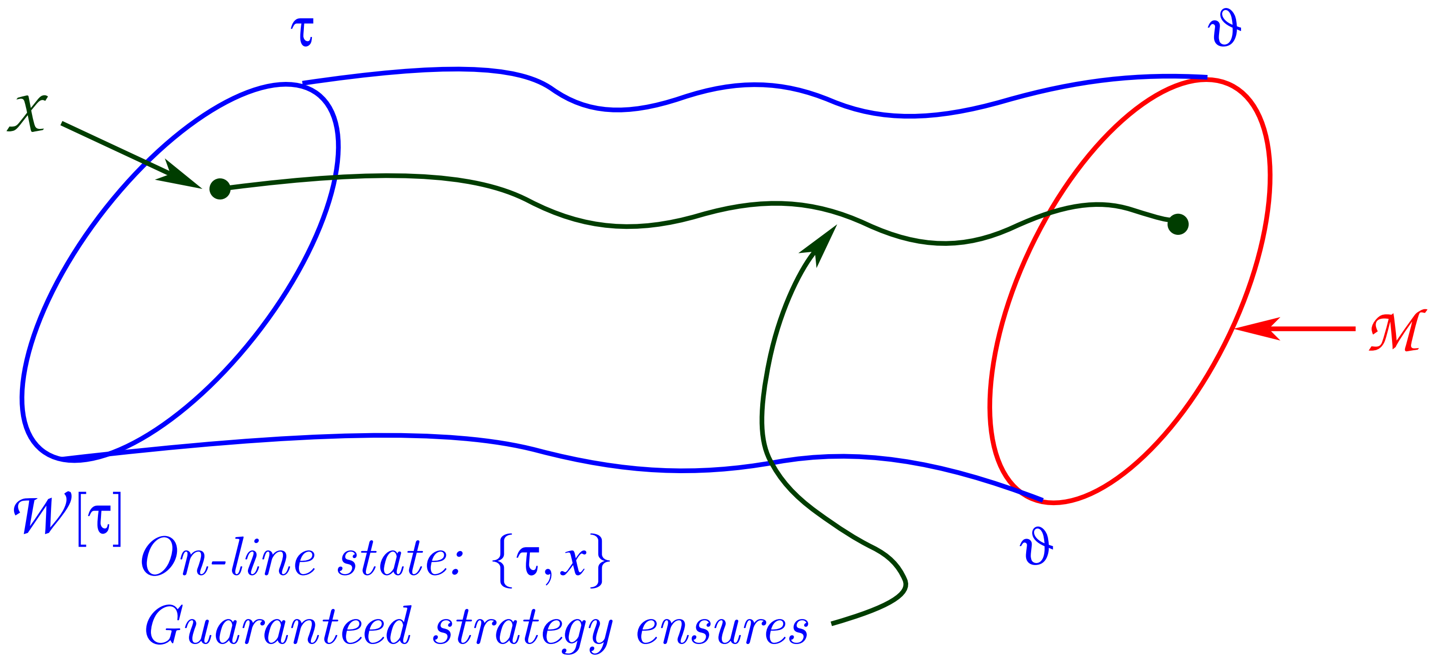
Here the control strategies are calculated by minimizing Hausdorff $h_+$ semi distance between// N.N.Krasovski's "aiming techniques."

**OTHER APPROACHES** deal through discretization of the continuous solutions or through discretizing the problem from the beginning.

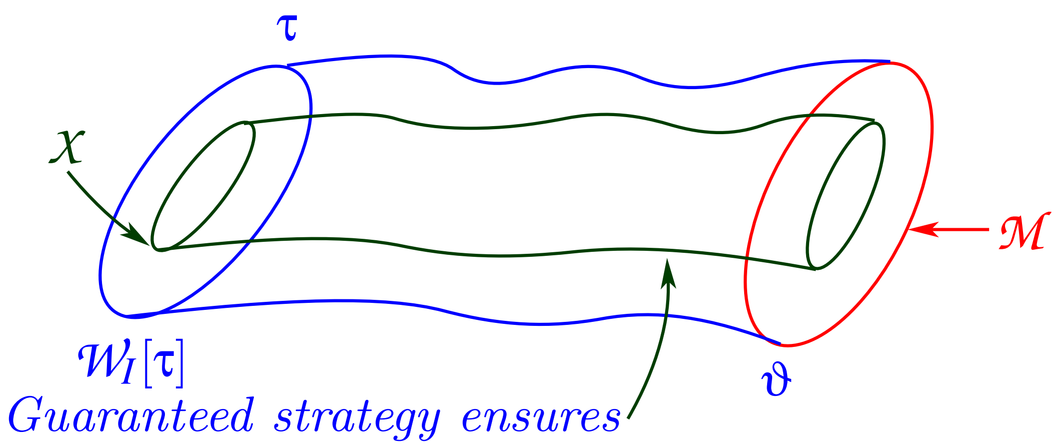
Comparative studies are necessary.

Can we plug in the controls found through discretization into the continuous equations and what will be the error ???

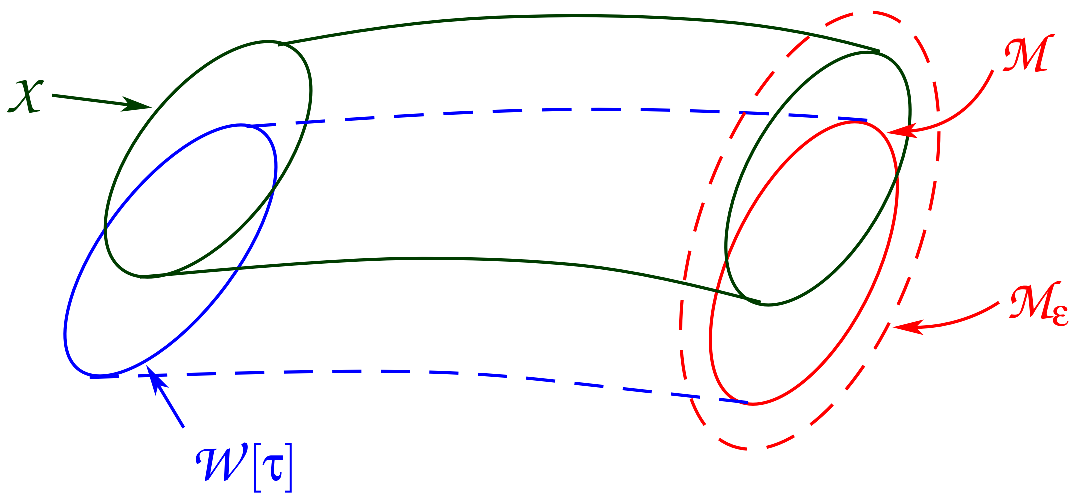
# *Complete measurements*



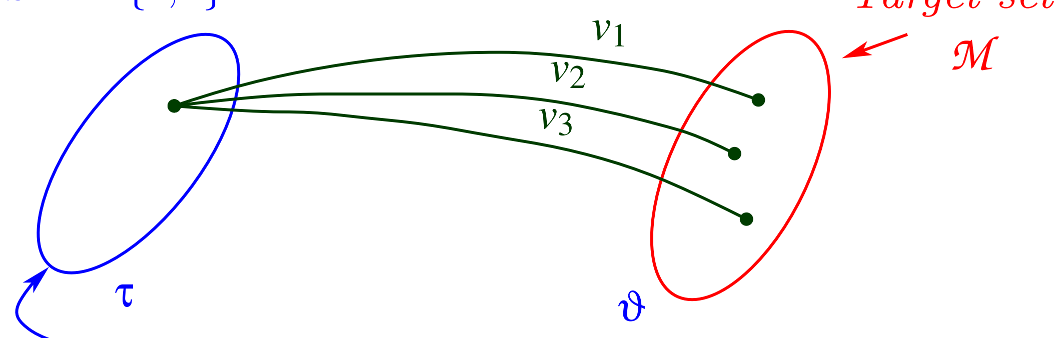
*Output feedback*



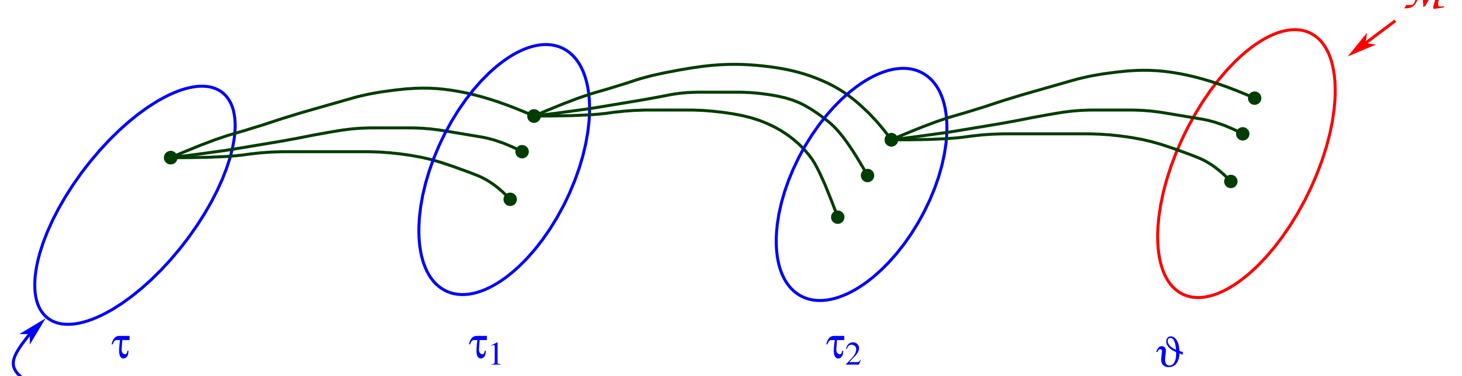

*If measurement error is too large*



# *Feedback control under complete measurements*




$\mathcal{W}[\tau] = T[\tau, \vartheta]\mathcal{M}$




$\mathcal{W}[\tau] = T[\tau, \tau_1]T[\tau_1, \tau_2]T[\tau_2, \vartheta]\mathcal{M}$

*limit case* $\mathcal{W}_N[\tau] \to \mathcal{W}[\tau], \quad N \to \infty, \quad \sigma_N = \max_i |\tau_{i+1} - \tau_i| \to 0$

$\mathcal{W}[\tau] \to$ *invariant set*

# Feedback control under Complete Measurements

STATE: $\{\tau, x\}$

INVARIANT SET: $\mathcal{W}[\tau] = \{x : \forall v(\cdot), \exists u(\cdot) \rightarrow x(\vartheta) \in \mathcal{M}\}$

CONTROLS: OPEN-LOOP

$$\text{Invariant set} \quad \mathcal{W}[\tau] = T[\tau, \vartheta]\mathcal{M}$$

This is a **linear map** $T : \mathcal{M}[\vartheta] \rightarrow \mathcal{W}[\tau]$, calculated through convex analysis

Under Matching Conditions:

$$\mathcal{P}(t) \equiv \alpha Q(t), \ \alpha \in (0,1)$$

TO FIND THE FEEDBACK CONTROL STRATEGY WE NEED THE INVARIANT SET !// THE CONTROL THEN ARRIVES FROM CONDITION:

$$\mathcal{U}(t,x) = \left\{ u : \max_v \left\{ \frac{d\mathrm{d}^2(x(\tau), \mathcal{W}[\tau])}{dt} \bigg|_{u,v} \leq 0 \right\} \right\}$$

NOTE THAT HERE WE HAVE TO SOLVE PROBLEMS IN FINITE TIME

THIS IS MUCH HARDER THAN SOLVING STABILIZATION PROBLEMS

(compare Value functions with control Liapunov functions: for linear-quadratic control problems in infinite time the Liapunov function is a value function)

The **invariant sets** $\mathcal{W}$ are
the backward reachability sets from set $\mathcal{M}$

If $\mathcal{W}[\tau]$ is calculated through open loop controls
$\mathcal{W}_c[\tau]$ is calculated through closed-loop controls, then
Under Matching Conditions we have:

$$\mathcal{W}[\tau] = \mathcal{W}_c[\tau]$$

Without Matching Conditions we have:

$$\mathcal{W}[\tau] \neq \mathcal{W}_c[\tau]$$

$\mathcal{W}[\tau] = T[\tau, \vartheta]\mathcal{M}$

$\mathcal{W}_N[\tau] = T[\tau, \tau_1] \ldots T[\tau_N, \vartheta]\mathcal{M}, \;\; \mathcal{W}_N[\tau] \to \mathcal{W}_c[\tau] (N \to \infty);$

## **Feedback control under** Complete Measurements

STATE: $\{\tau, x\}$   NO matching conditions:

Here $\tau_i$ are the points of DISCRETE MEASUREMENTS.

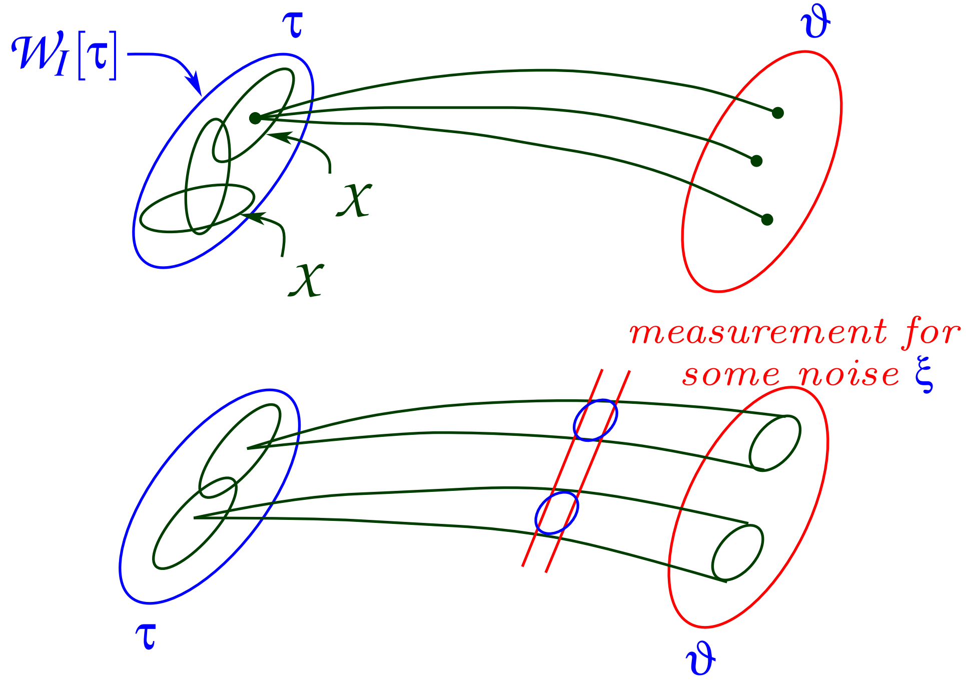
$\mathcal{W}_N[\tau] = T[\tau, \tau_1] \ldots T[\tau_N, \vartheta]\mathcal{M}$   CONTROLS: piecewise open-loop

(compare with model-predictive controls)

Limit case (with $N \to \infty$, $\sigma_N = \max|\tau_{i+1} - \tau_i| \to 0$) :

$\mathcal{W}_N[\tau] \to \mathcal{W}_c[\tau]$ – INVARIANT SET

# *Output Feedback Control*
## *(Incomplete Measurements)*



## **Output Feedback Control** Incomplete Measurements

STATE: $\{\tau, \mathcal{X}[\tau]\}$,

### Under Matching Conditions

Invariant Set:

$\mathcal{W}_I[\tau] = \{\mathcal{X} : \forall \in \mathcal{X}, \forall v(\cdot), \; \exists u(\cdot) \; x(\tau) \to x(\vartheta) \in \mathcal{M}_{sub} \subseteq \mathcal{M}\}$ under on-line state constraint

$$y(t) - G(t)x(t) + \xi(t) \in \mathcal{R}(t)$$

Calculated through CLOSED-LOOP CONTROLS

$$\mathcal{W}_I[\tau] = \cup\{\mathcal{X}\}, \quad \mathcal{X} = x + \Omega$$

with $\quad \mathcal{W}_I[\tau] = T[\tau, \vartheta]\mathcal{M}$

*** **Output Feedback Control** Incomplete Measurements

STATE: $\{\tau, \mathcal{X}[\tau]\}$,   NO matching conditions

DISCRETE measurements: $\mathcal{W}_{IN}[\tau] = T_I[\tau, \tau_1] \ldots T_I[\tau_N, \vartheta]\mathcal{M}$

CONTINUOUS measurements: Limit case (with $N \to \infty$, $\sigma_N \to 0$)

$\mathcal{W}_{IN}[\tau] \to \mathcal{W}_{Ic}[\tau]$ – INVARIANT SET

FROM PIECEWISE-CONTINUOUS SOLUTIONS TO FEEDBACK CONTROL SOLUTION STRATEGY

## REDUCTION to FINITE-DIMENSIONAL SCHEMES

To find feedback controls we need to calculate

$d(x, \mathcal{W}[t])$ – under complete measurements (finite-dimensional scheme)

$h_+(\mathcal{X}[t], \mathcal{W}_I[t])$ – under incomplete output measurements

(in general - an infinite-dimensional scheme)

For linear systems with convex constraints we have:

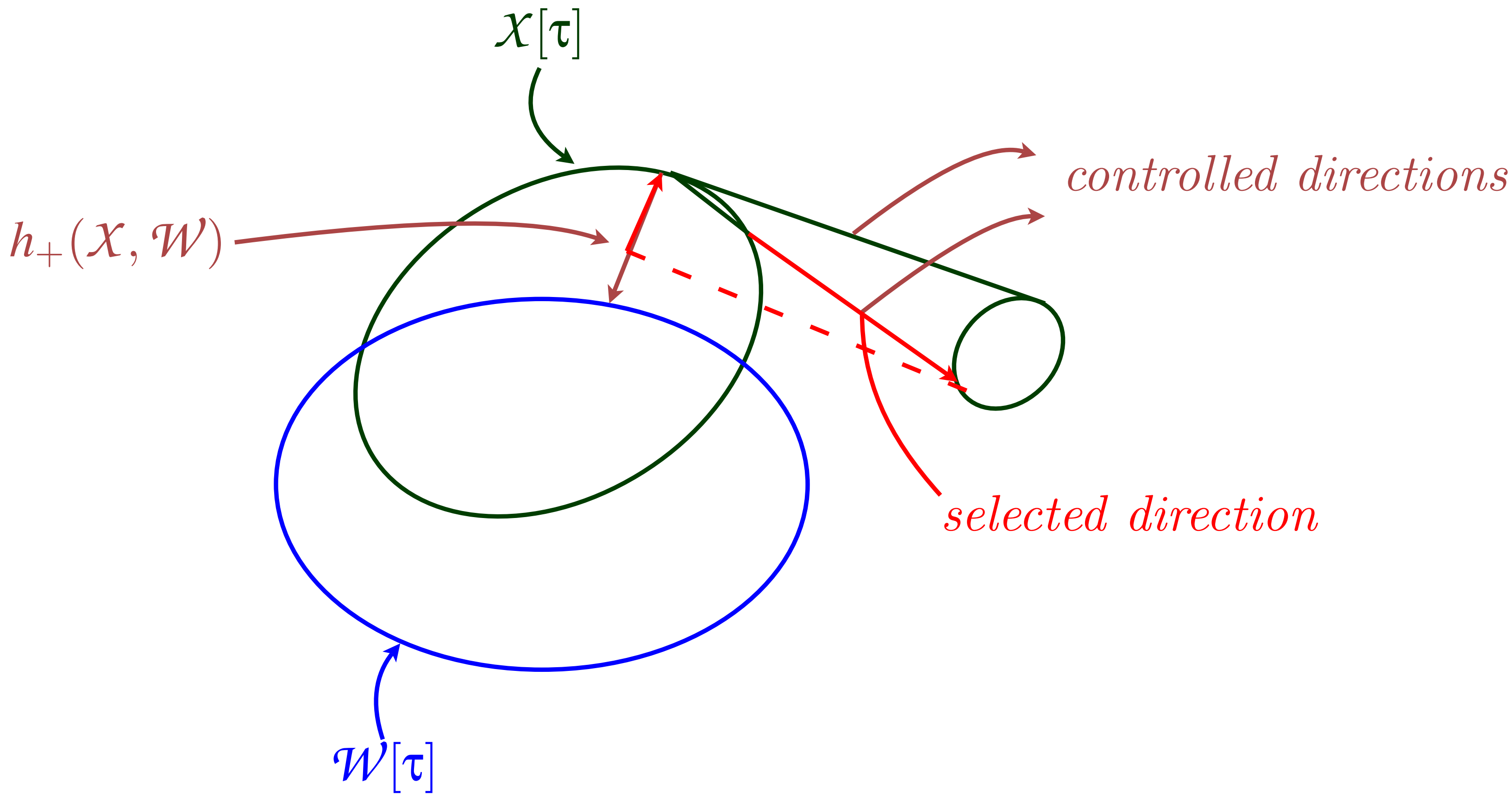
$$\mathcal{W}_{Ic}[t] = \mathcal{W}_c[t]$$

Then, instead of $h_+(\mathcal{X}[t], \mathcal{W}_{Ic}[t])$ we need $h_+(\mathcal{X}[t], \mathcal{W}_c[t])$

Under Matching Conditions :

$$\mathcal{W}_{Ic}[t] = \mathcal{W}_c[t] = \mathcal{W}[t]$$

This may be exactly calculated through finite-dimensional schemes




*Selection of control strategy*

$$\mathcal{U}(\tau,x)=\left\{u\ :\ \frac{d\,h_+(\mathcal{X}[\tau],\mathcal{W}[\tau])}{d\tau}\leq 0\right\}$$

## EQUATIONS FOR THE SYNTHESIZED SYSTEM

The state of the system is $\{\mathcal{X}[t] = x^*(t) + \Omega[t]\}$

We use the support function for set

$$\Omega[t] \;:\; \varphi(t,l) = \rho(l \mid \Omega[t]) = \max\{(l\,,x) \mid x \in \Omega[t]\}$$

The evolution equations for $x^*(t), \Omega[t]$ are

$$\begin{cases} \dot{x}^* = B(t)\,\mathcal{U}(t,\mathcal{X}), \\ \dfrac{\partial\rho(l \mid \Omega[t])}{\partial t} = \Psi(t,l,\Omega[t],\mathcal{Z}[t]) \end{cases}$$

This a PDE for the support function $\partial\rho(l \mid \Omega[t])$.

($\mathcal{Z}[t]$ is the measurement set).

IF measurements are DISCRETE, at instants $\{\tau_{\mathbf{i}}\}$, then

$$\Omega[\tau_{i+1}] = \Omega[\tau_{i+1} - 0] \cap \mathcal{Z}[\tau_{i+1}]$$

$$\partial\rho(l \mid \Omega[t])/\partial t = \rho(l|B(t)\mathcal{Q}(t)),\ t \in [\tau_i, \tau_{i+1}),\ \Omega[\tau_0] = \mathcal{X}^0.$$

Between measurements we calculate ”ordinary” reach set
without state constraints.
Support function $\rho(l \mid \Omega[t])$ may be calculated **exactly** through
Duality Theory of Convex Analysis

But we need **effective calculation** for Large Dimensions

**This can be reached through**

ELLIPSOIDAL or POLYHEDRAL CALCULUS !

# III. The Solution Through Ellipsoidal Techniques

An ellipsoid $(P > 0)$

$$\mathcal{E}(p,P) = \{x : (x-p, P^{-1}(x-p)) \leq 1\}$$

Its support function

$$\rho(l|\mathcal{E}(p,P) = (l,x) + (l,Pl)^{1/2}$$

The target set $\mathcal{M} = \mathcal{E}(m,M)$

Hard bounds:

$$x(t_0) \in \mathcal{E}(x^0, X^0),\ u \in \mathcal{E}(p(t), P(t)),$$

$$f(t) \in \mathcal{E}(q(t), Q(t)),\ \xi(t) \in \mathcal{E}(0, R(t)).$$

Here $M = M' > 0$, $X^0 = X^{0'} > 0$, and

$$\mathcal{P}(t) = \mathcal{P}(t)' > 0,\ \mathcal{Q}(t) = \mathcal{Q}(t)' > 0,\ \mathcal{R}(t) = \mathcal{R}'(t) > 0.$$

**Stage 1**. Solve Problem GSE : find information set $\mathcal{W}[\tau]$.

This actually is the reach set of system

$$\dot{w} \in C(t)\mathcal{E}(q(t), Q(t)),\ w(t_0) \in \mathcal{E}(x^0, X^0),$$

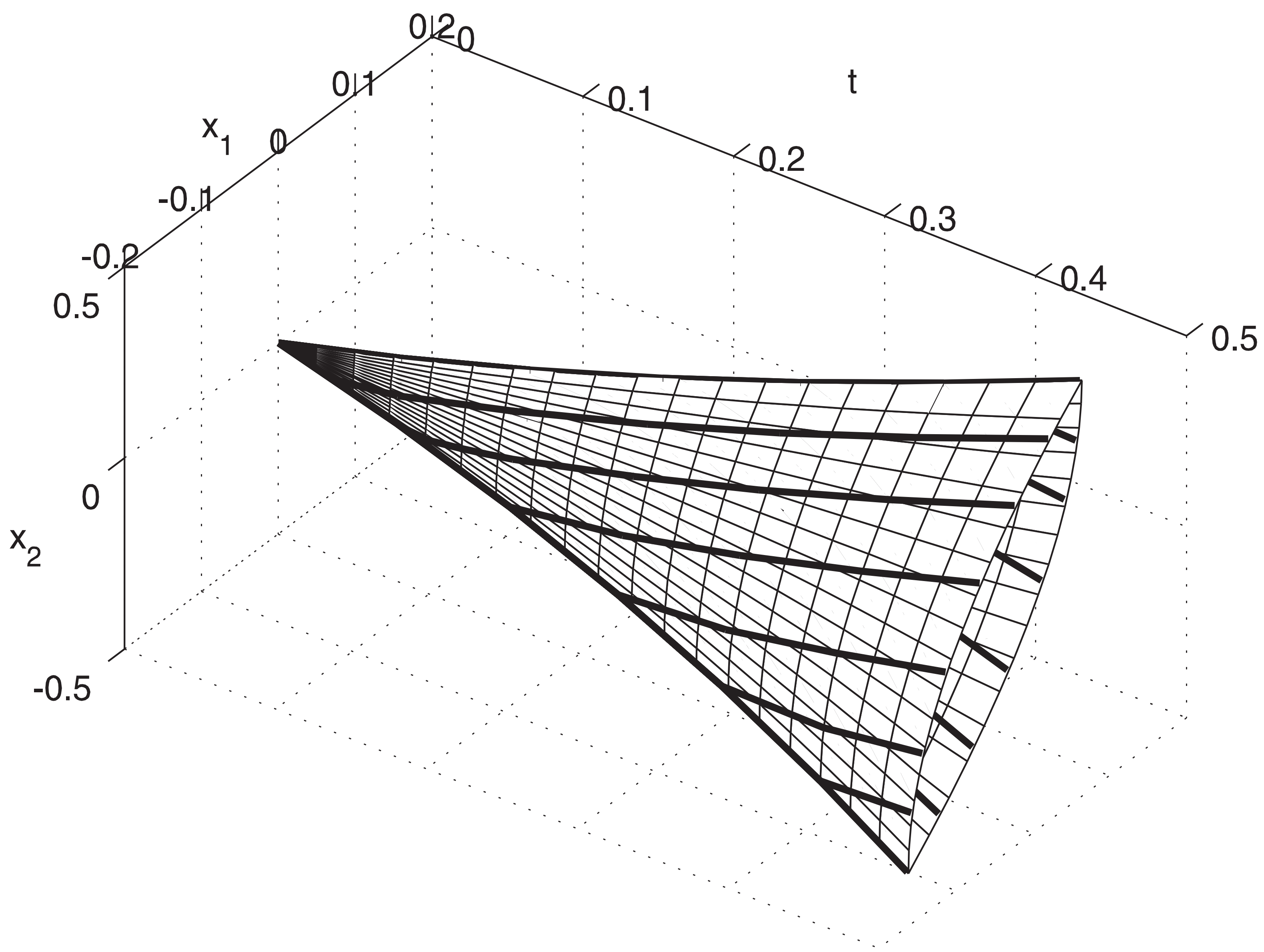
under on-line state constraint

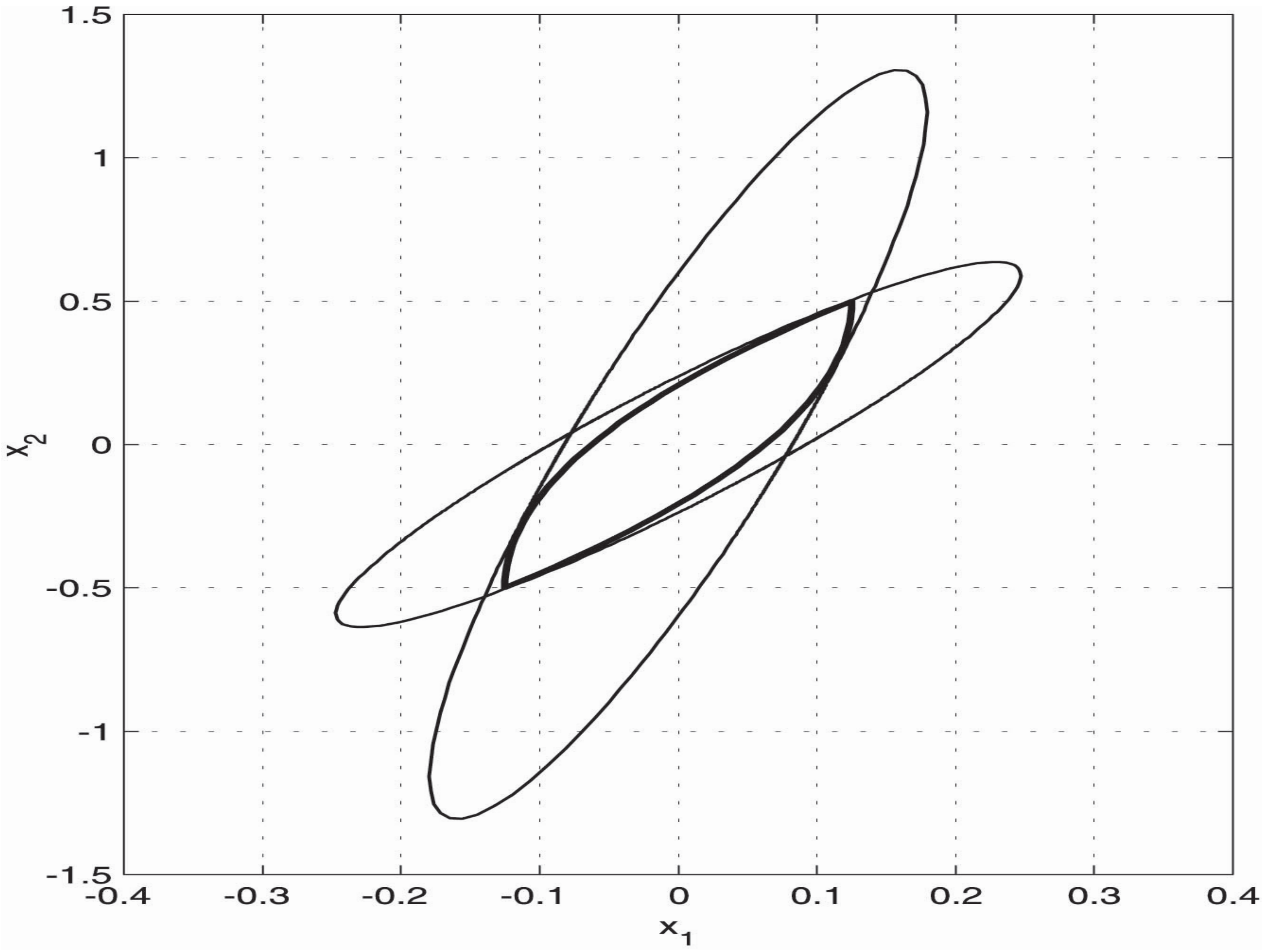
$$z^*(t) - H(t)w(t) \in \mathcal{E}(0, R(t)),\ t \in [t_0, \tau],$$

with given $z^*(t)$ .

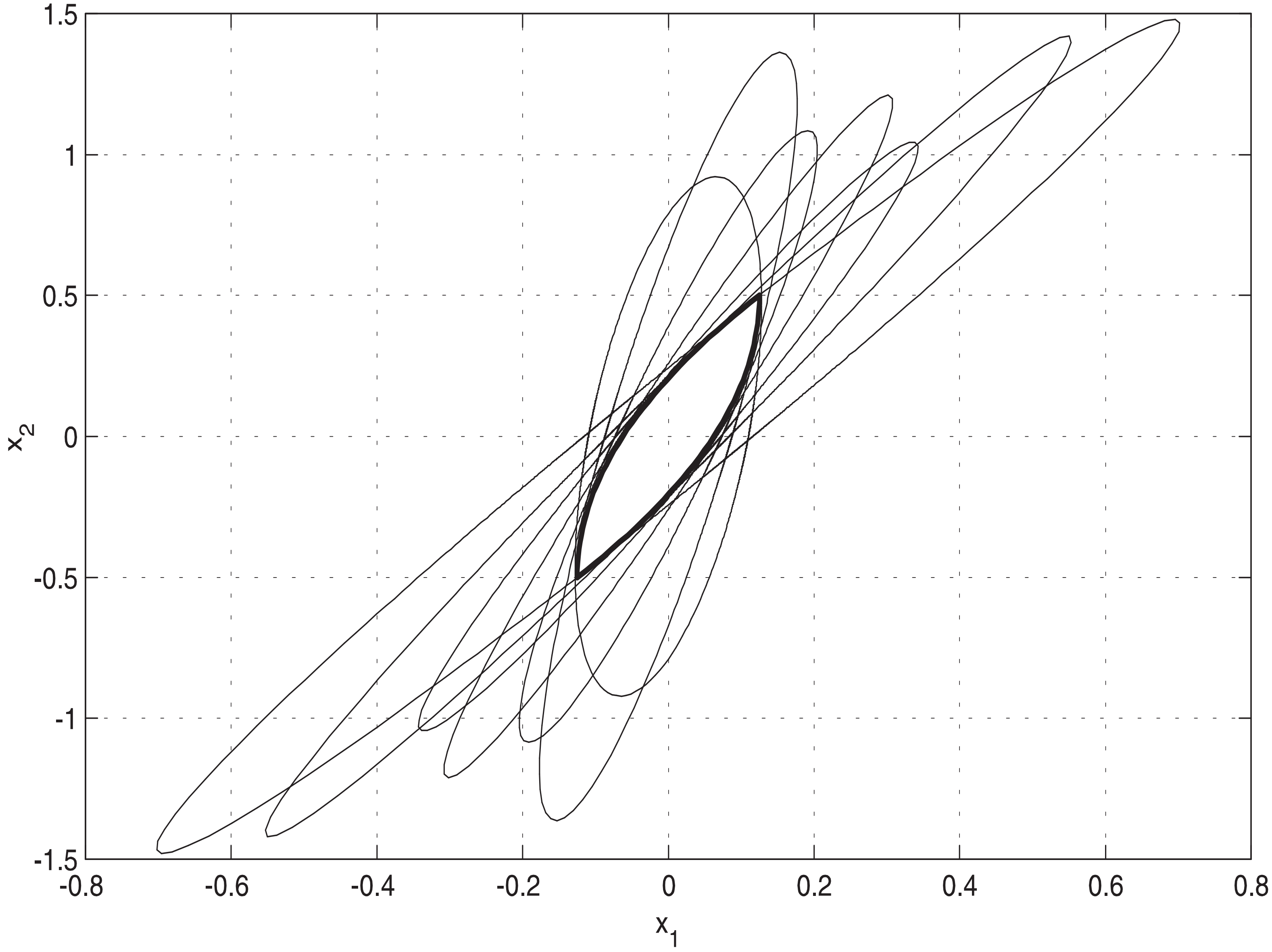
We present $\mathcal{W}[\tau]$ through parametrized
external ellipsoids!

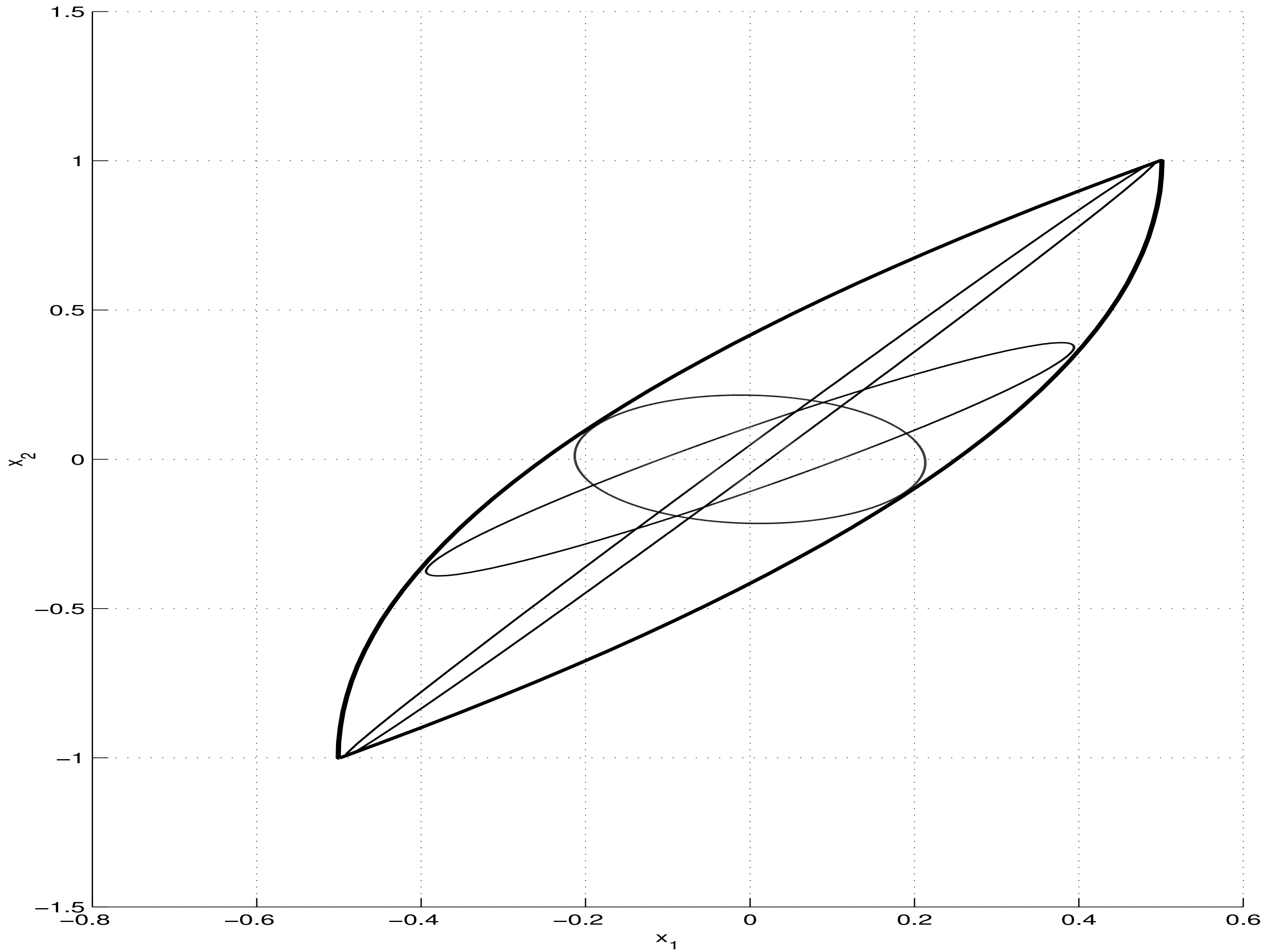
$$\mathcal{W}[\tau] \subseteq \mathcal{E}(w_+(\tau), W_+(\tau) \mid \omega(\tau)),$$

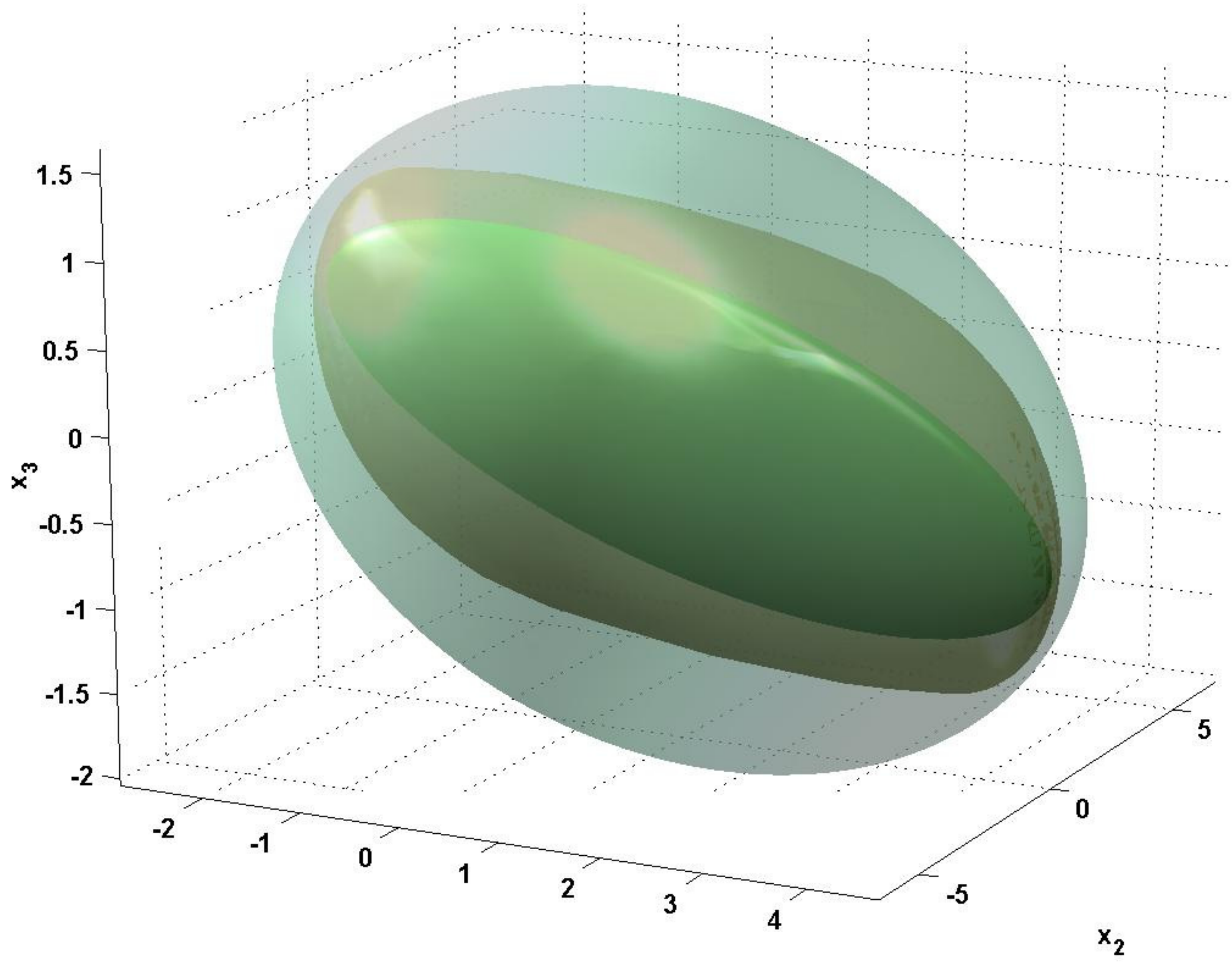

0.2
0
0.1
t
x_1
0
0.1
-0.1
0.2
0.3
-0.2
0.4
0.5
0.5
0
x_2
-0.5

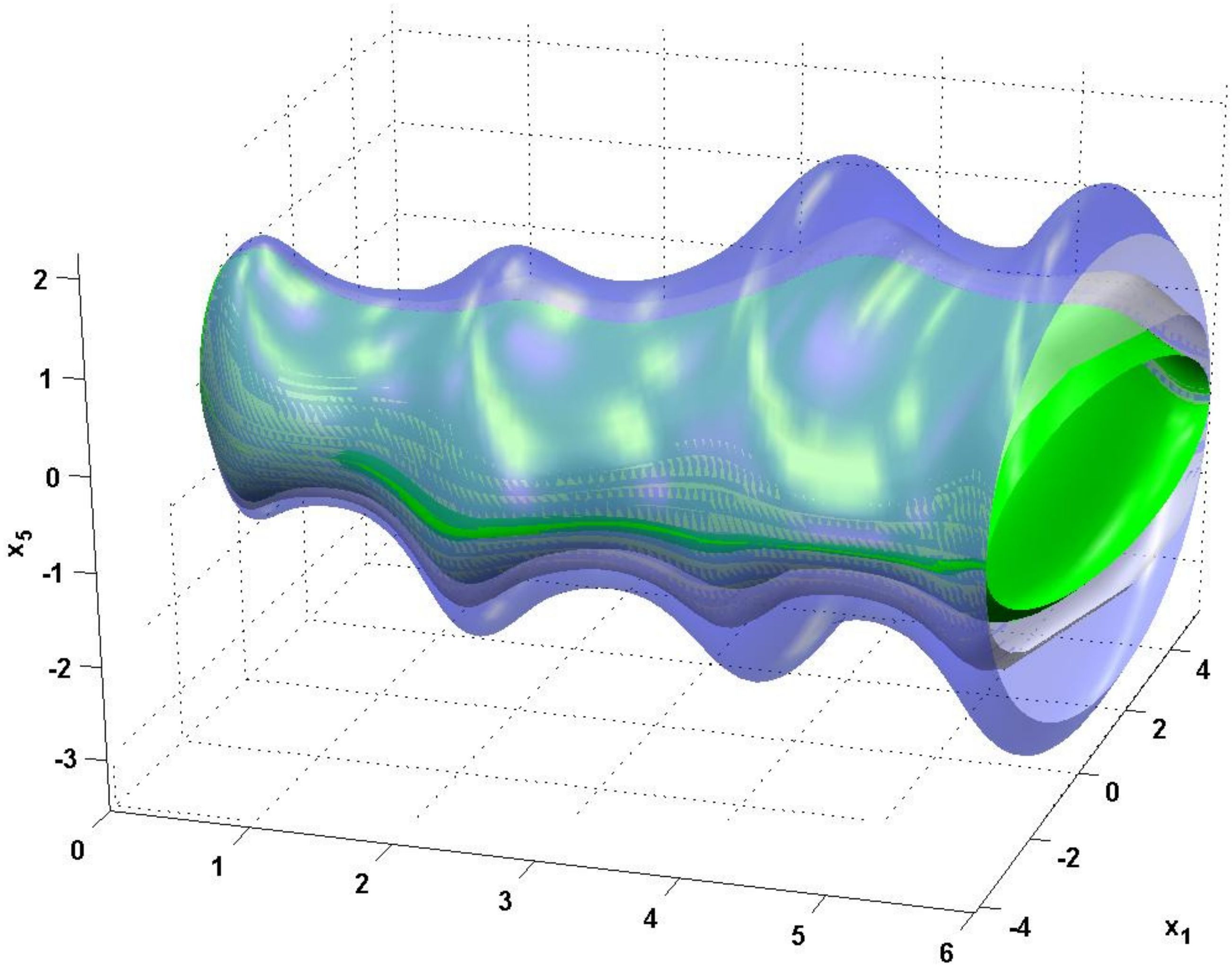


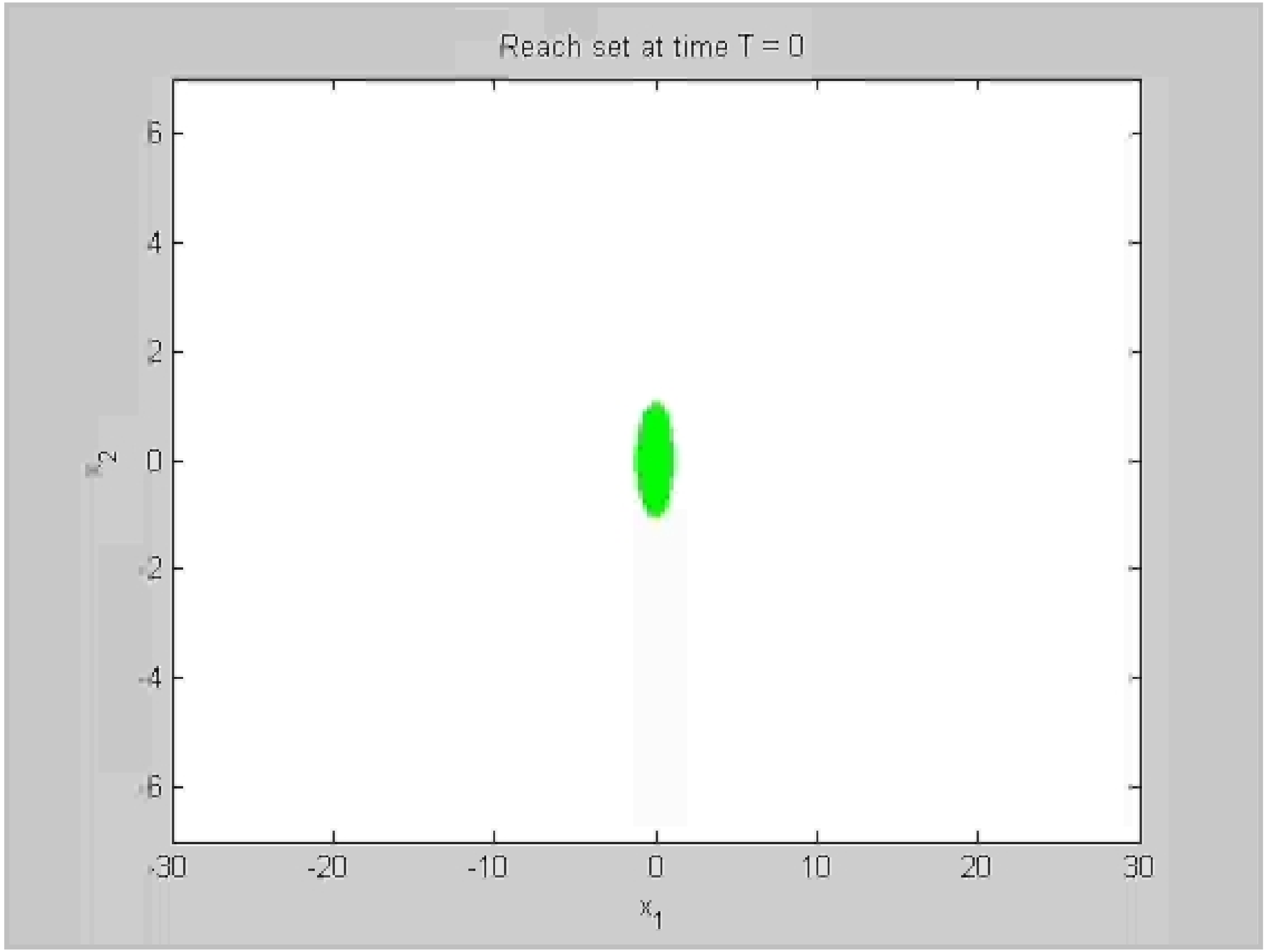
1.5
1
0.5
0
-0.5
-1
-1.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
$x_1$
$x_2$

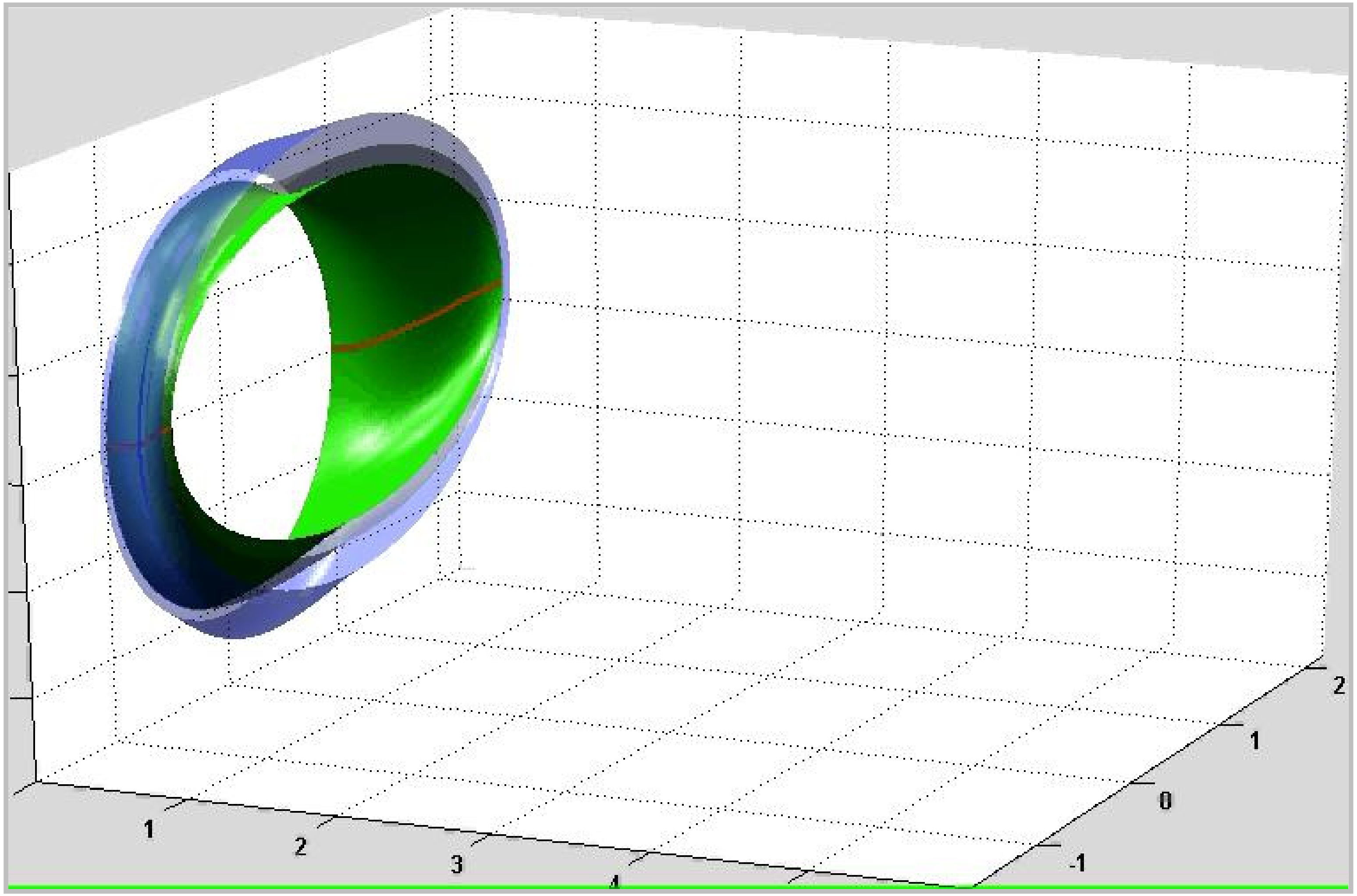

1.5
1
0.5
0
-0.5
-1
-1.5
$x_2$
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
$x_1$

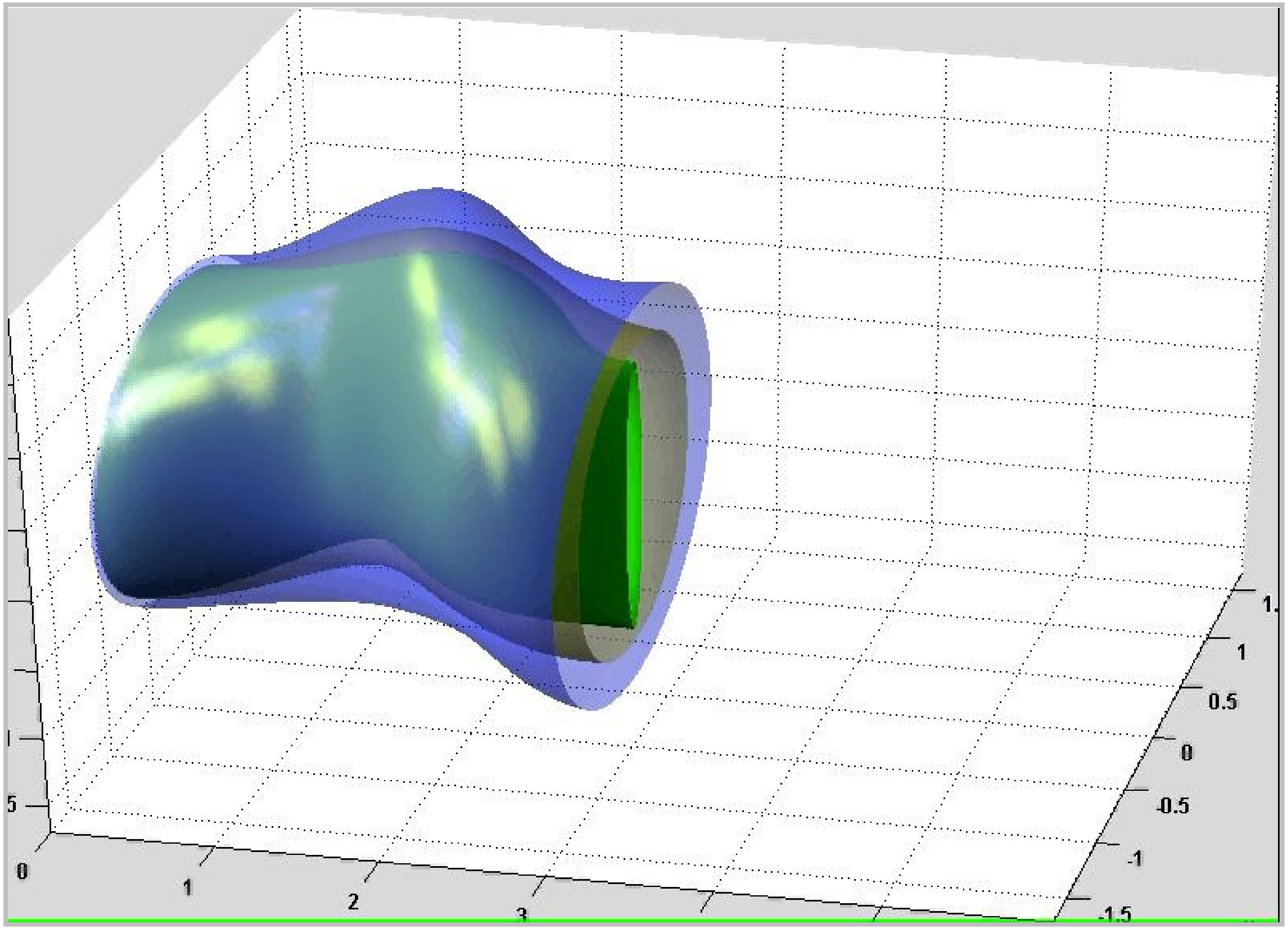

1.5
1
0.5
0
−0.5
−1
−1.5
$x_2$
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
$x_1$

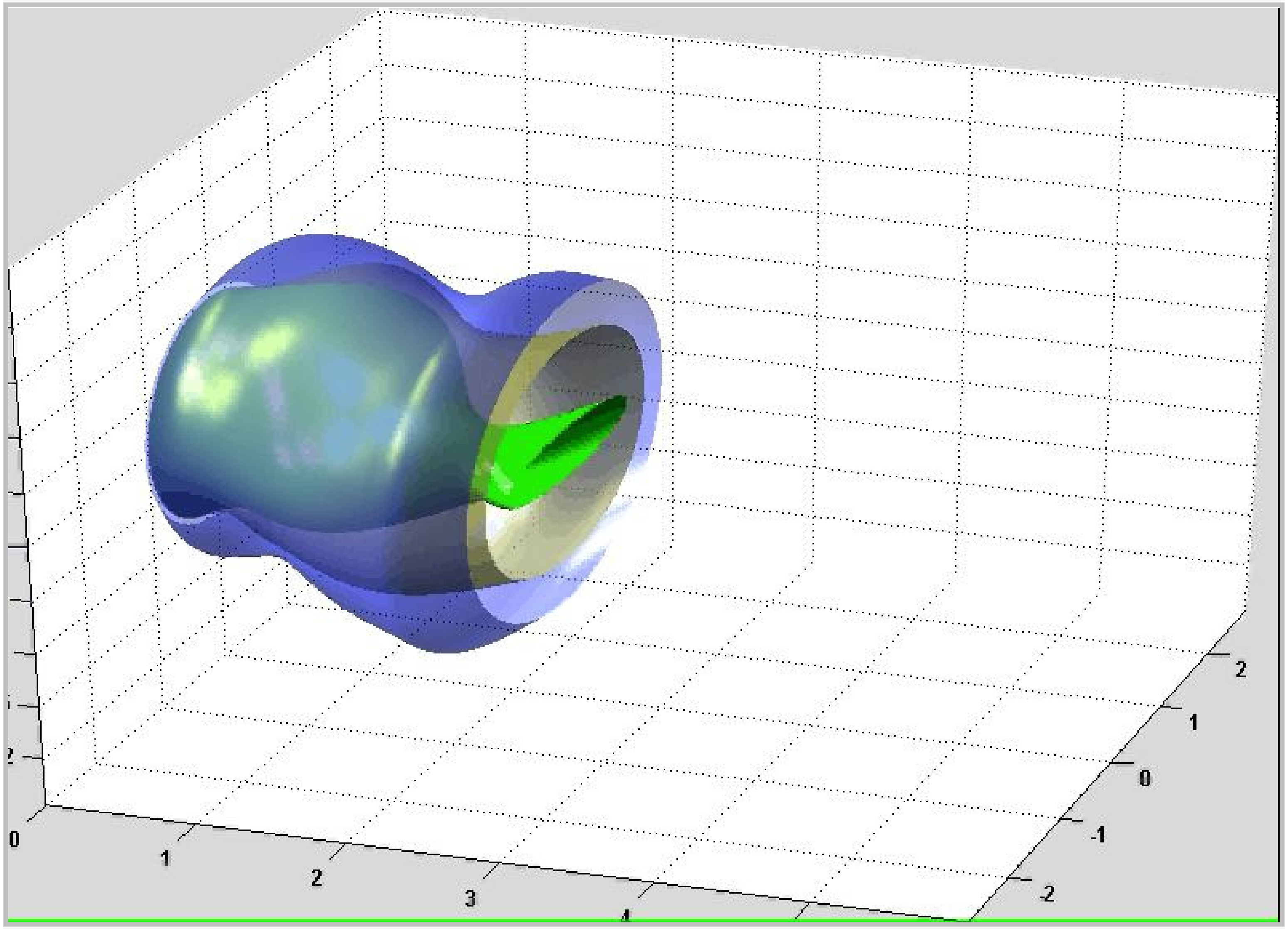

1.5
1
0.5
0
-0.5
-1
-1.5
-2
x3
-2
-1
0
1
2
3
4
-5
0
5
x2

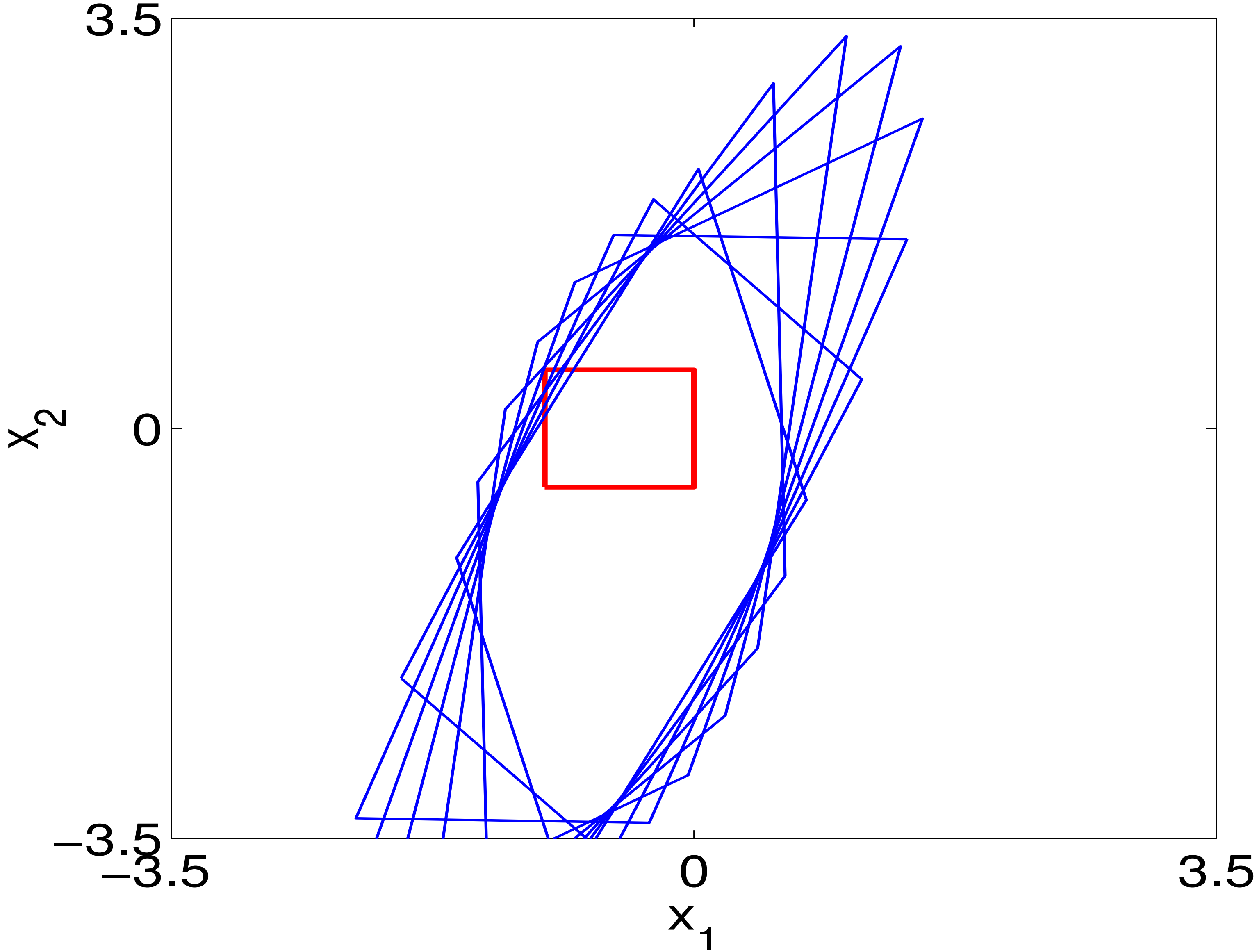

2
1
0
-1
-2
-3
$x_5$
0
1
2
3
4
5
6
-4
-2
0
2
4
$x_1$

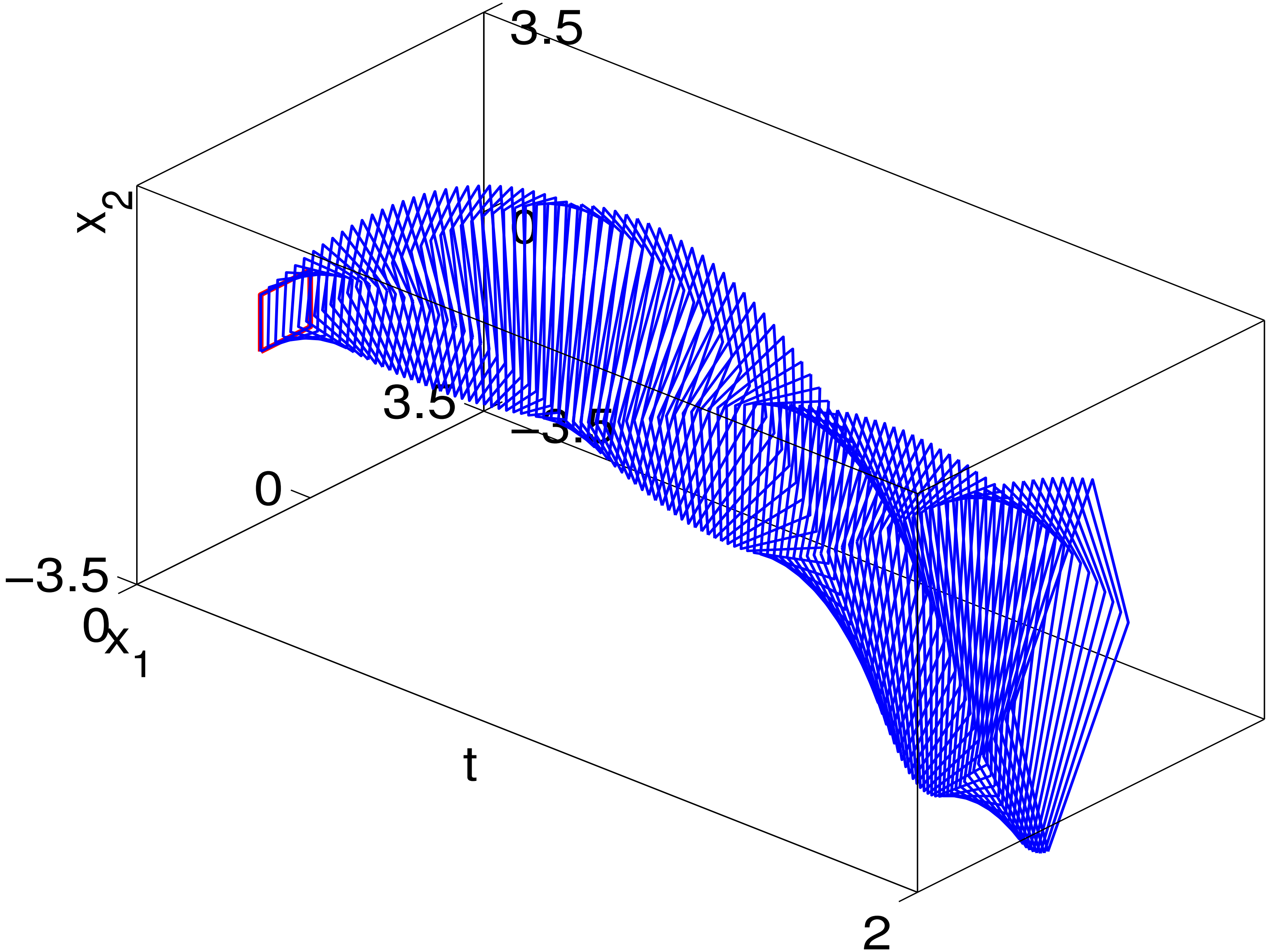

Reach set at time T = 0
6
4
2
0
-2
-4
-6
-30
-20
-10
0
10
20
30
$x_1$
$x_2$


1
2
3
4
-1
0
1
2

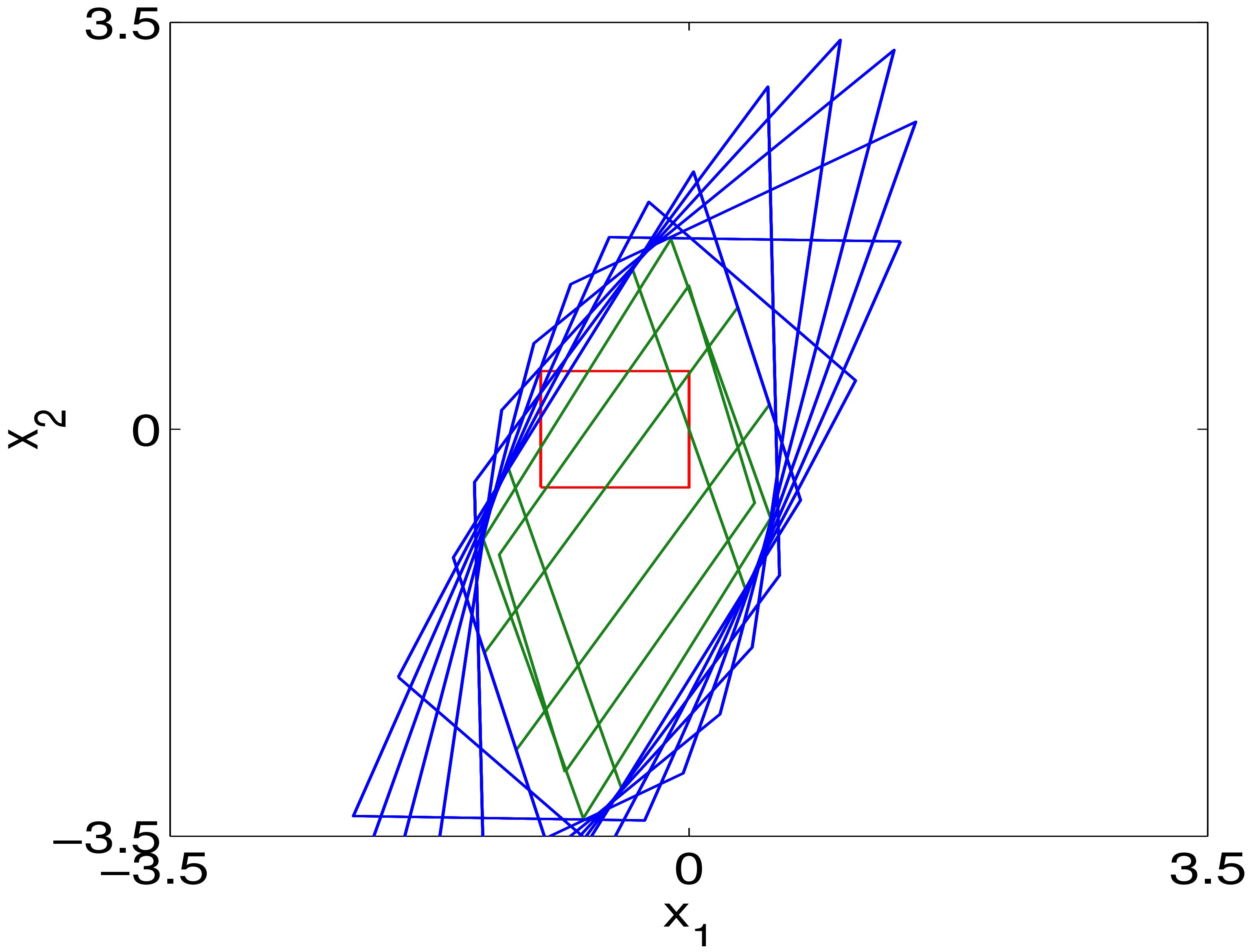

0
1
2
3
1.
1
0.5
0
-0.5
-1
-1.5
5

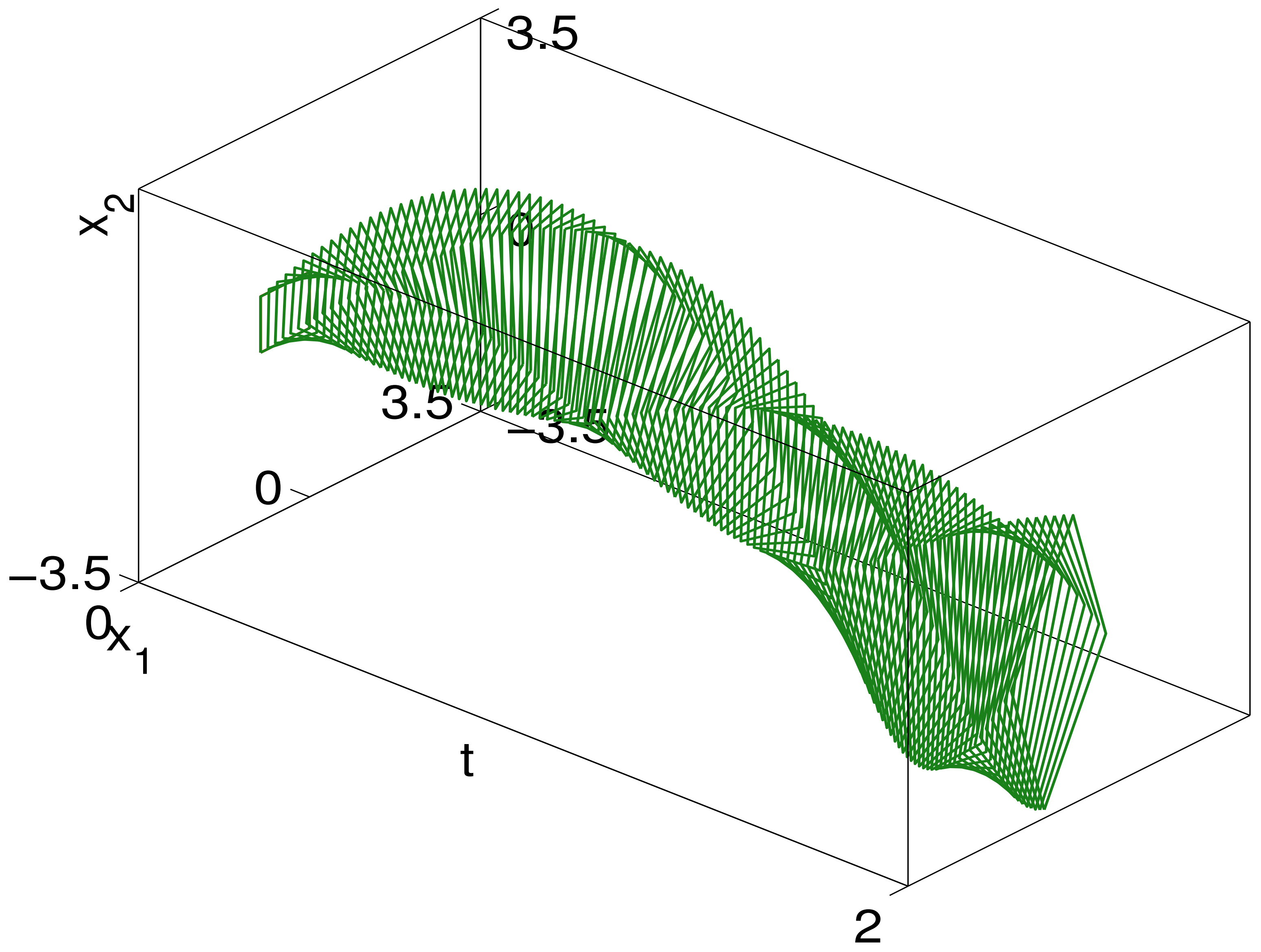

0
1
2
3
2
1
0
-1
-2


3.5
0
−3.5
−3.5
0
3.5
x1
x2

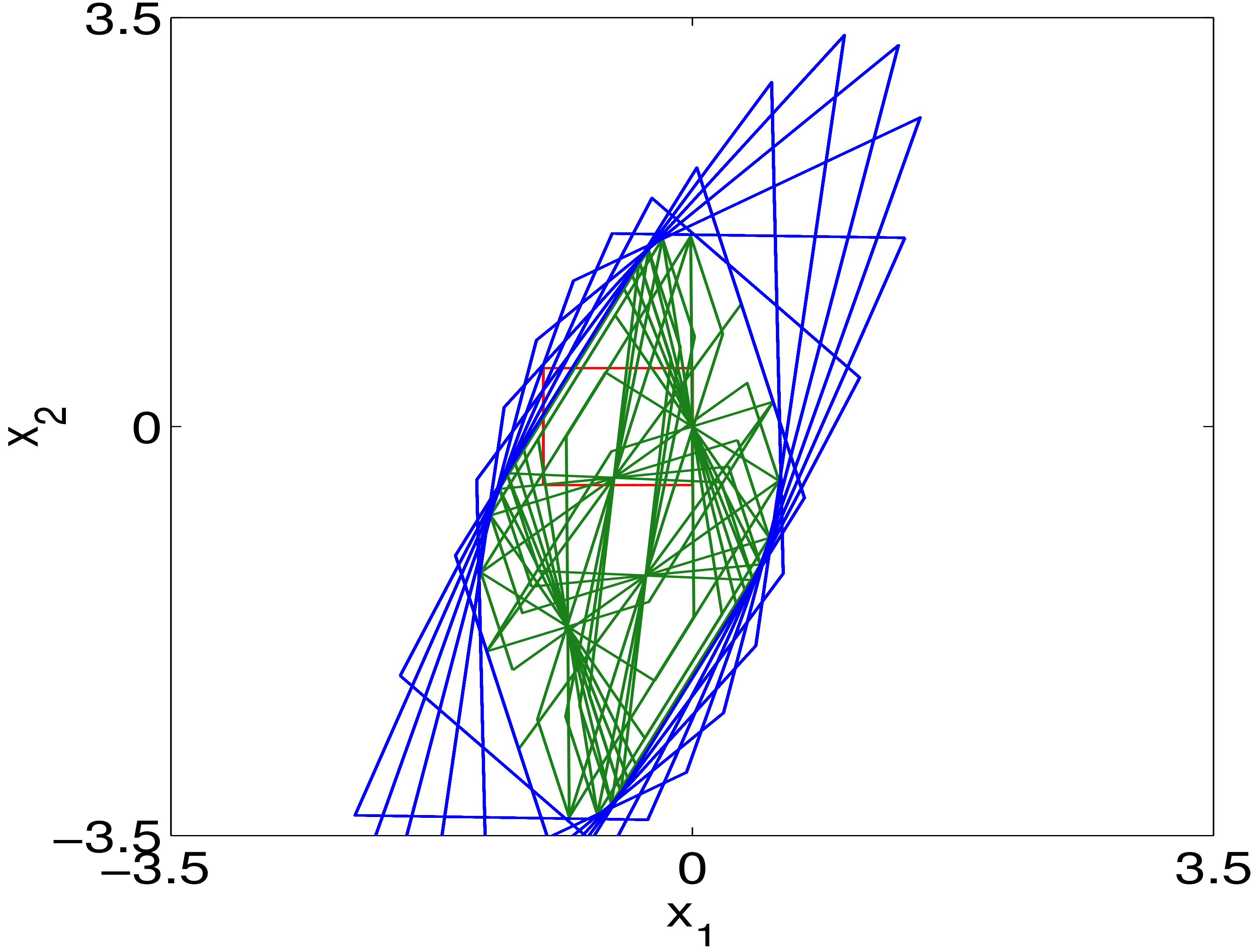


x2
3.5
0
−3.5
3.5
0
−3.5
0
x1
t
2


3.5
0
−3.5
$x_2$
−3.5
0
3.5
$x_1$



3.5
x2
0
3.5
-3.5
0
-3.5
0
x1
t
2


3.5
0
−3.5
x2
−3.5
0
3.5
x1

# Exact ellipsoidal representations of $\mathcal{X}[\tau]$

Denote parameters

$$\chi_+(\tau) = \{\gamma_u(\cdot), S(\cdot)\}, \;\; \chi_-(\tau) = \{\gamma_f(\cdot), S_1(\cdot), S_2(\cdot)\}$$

$$\mathcal{E}(x_e, X_+(\tau)) = \mathcal{E}(x_e, X_+(\tau); \chi_+(\tau)), \;\; \mathcal{E}(x_e, X_-(\tau)) = \mathcal{E}(x_e, X_-(\tau); \chi_-(\tau))$$

Then we have

**Theorem.** The following representation is true

$$\bigcup \left\{ \mathcal{E}(x_e, X_-(\tau)) \middle| \chi_-(\tau) \right\} = \mathcal{X}^*[\tau] = \bigcap \left\{ \mathcal{E}(x_e, X_+(\tau)) \middle| \chi_+(\tau) \right\},$$

To Specify control strategy $U^0(\tau, \mathcal{X}, \mathcal{W})$
we need value function

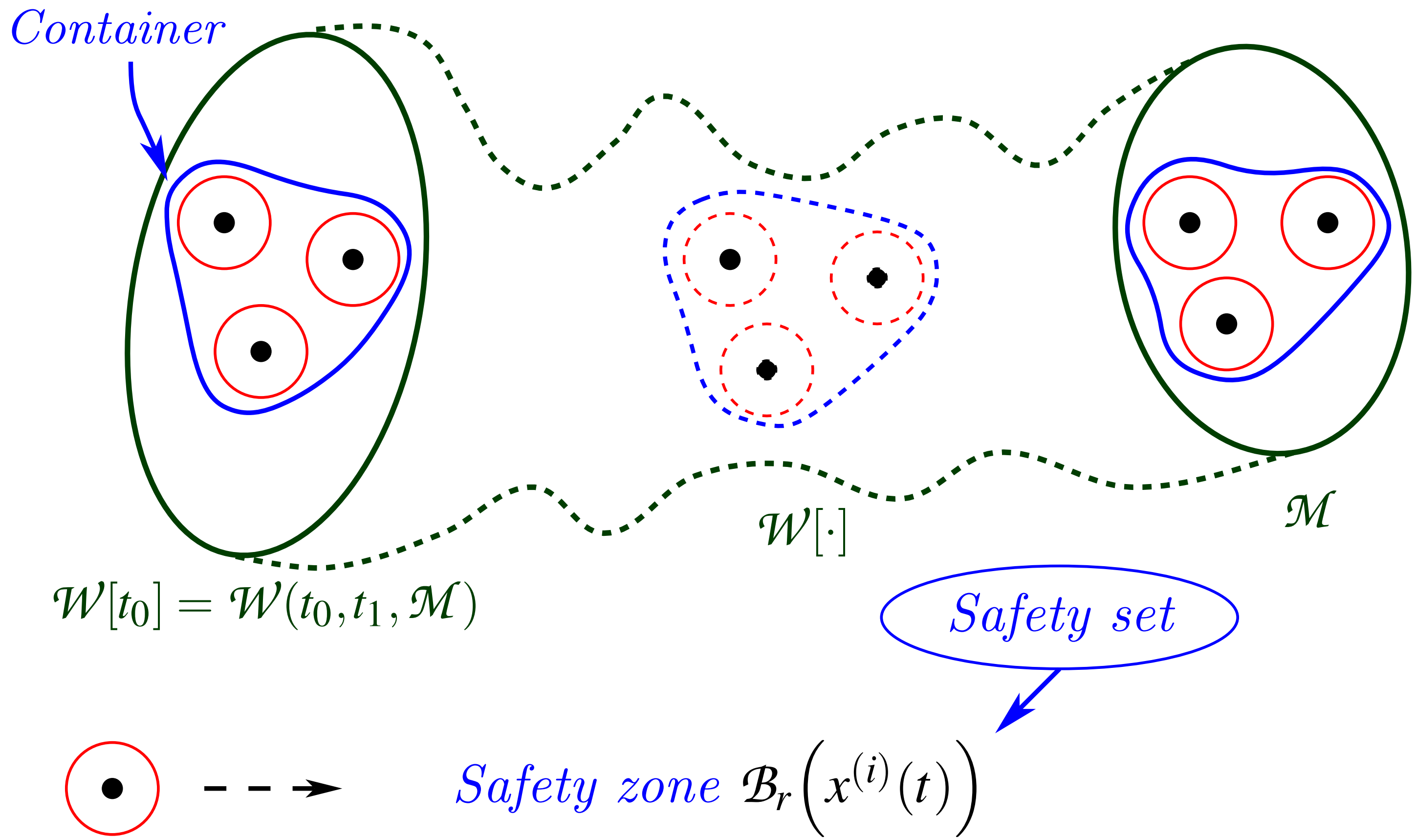
$$\mathcal{V}(\tau, \mathcal{X}, \mathcal{W}) = h_+(\mathcal{X}, \mathcal{W})$$

but use an ellipsoidal approximation

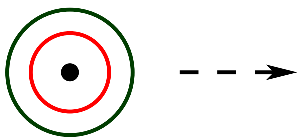
$$\mathcal{V}_{EL}(\tau, \mathcal{X}, \mathcal{W}) = d(\mathcal{E}(x(\tau), X_+(\tau)), \mathcal{E}_( w(\tau), W_-(\tau))$$

$$\mathcal{X}(\tau) \subseteq \mathcal{E}(x(\tau), X_+(\tau)),\ \mathcal{W}[\tau] \supseteq \mathcal{E}_( w(\tau), W_-(\tau)$$

## *Team Control Synthesis*
## *Complete measurements*

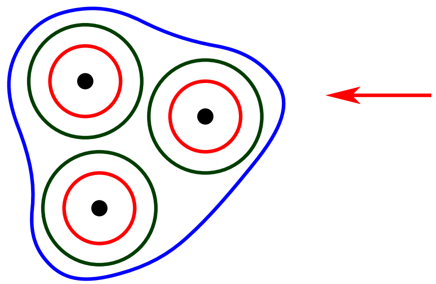


# Incomplete measurements



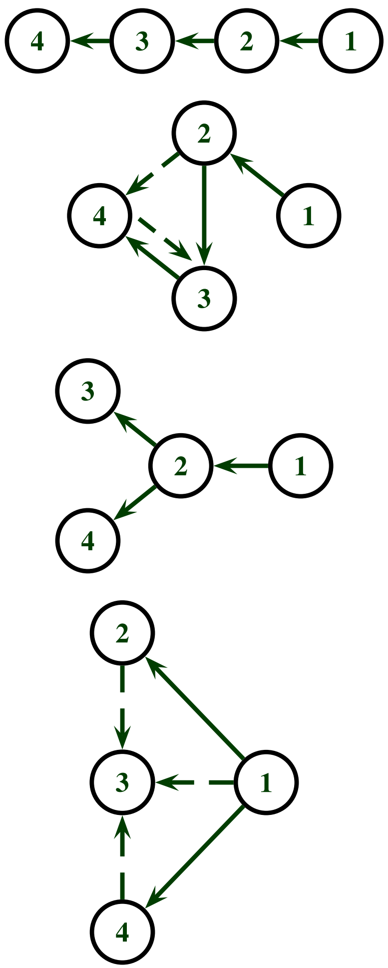
Safety zone:

$$\underbrace{\mathcal{B}_r}_{\text{Safety set}}\left(x^{(i)}(t)\right) + \underbrace{\mathcal{X}^{(i)}(t)}_{\text{information set}} = \underbrace{\mathcal{X}_{\mathcal{B}}^{(i)}(t)}_{\text{Total safety set}}$$



$\mathcal{X}_{\mathcal{B}}^{(i)}(t)$

# Team Control Synthesis



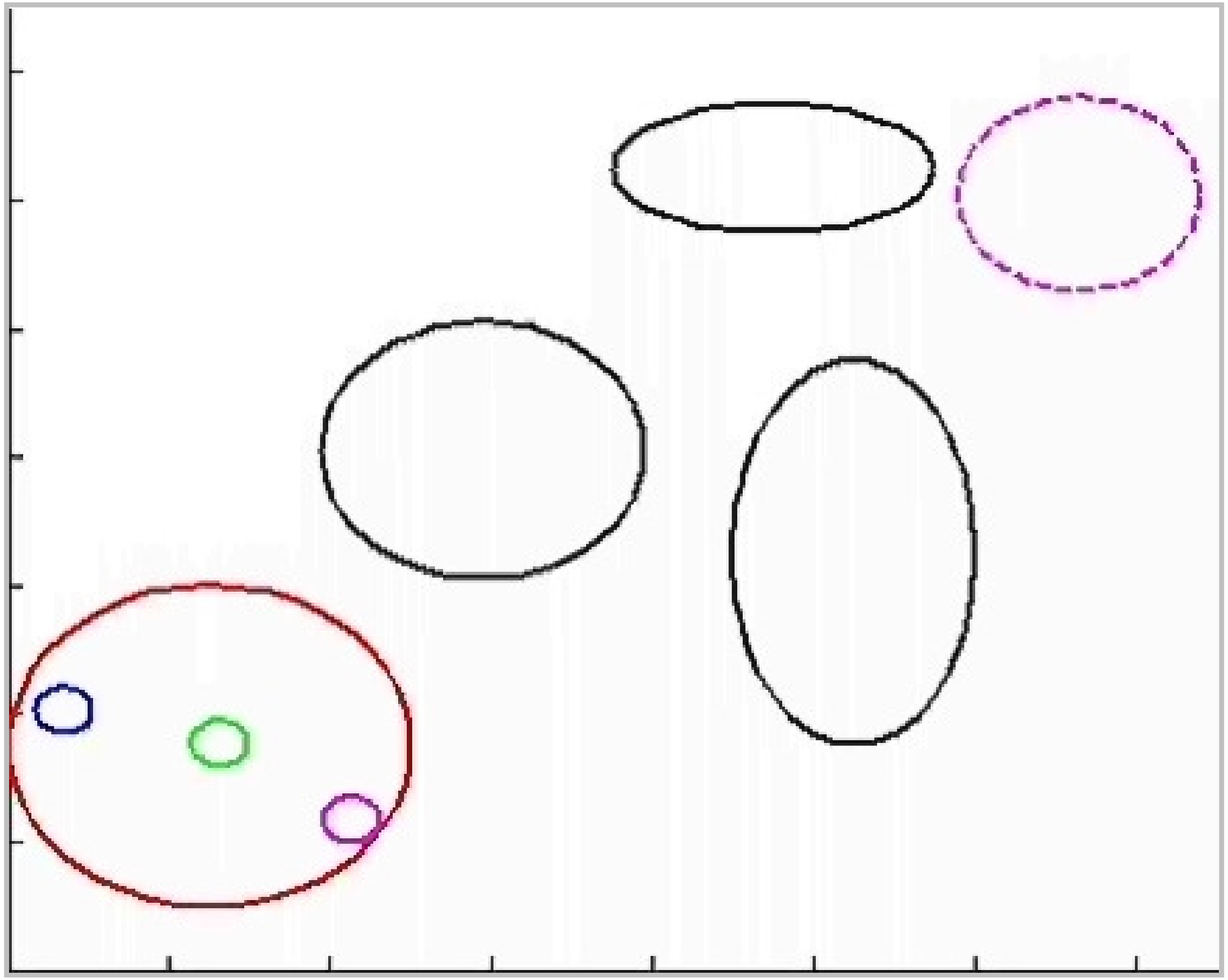
$$\begin{cases} y_1 = Gx^{(1)} + \xi_1 \\ y_2 = (x^{(2)} - x^{(1)}) + \xi_2 \\ y_3 = (x^{(3)} - x^{(2)}) + \xi_3 \\ y_4 = (x^{(4)} - x^{(3)}) + \xi_4 \end{cases}$$

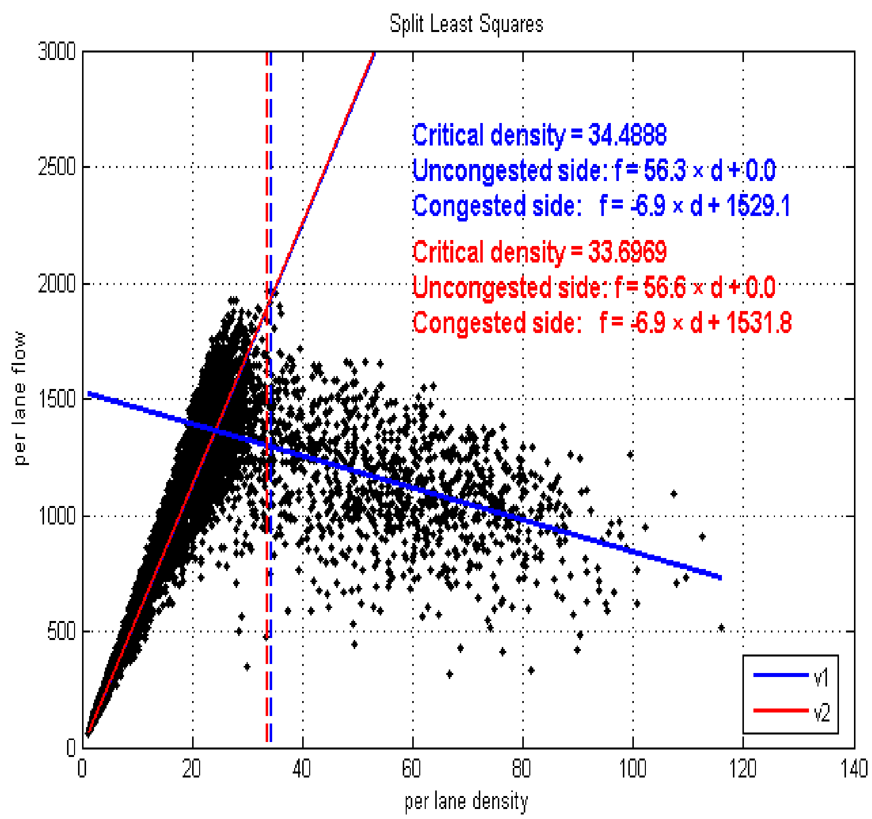
$$\begin{cases} \|\xi_1\| \leq \mu_1 \\ \|\xi_2\| \leq \mu_2\|x^{(2)} - x^{(1)}\| \\ \|\xi_3\| \leq \mu_3\|x^{(3)} - x^{(2)}\| \\ \|\xi_4\| \leq \mu_4\|x^{(4)} - x^{(3)}\| \end{cases}$$

$$\begin{cases} \mathcal{X}^{(1)}(t) \\ \mathcal{X}^{(2)} = \mathcal{X}^{(1)}(t) + \underline{\mathcal{X}_{21}(t)} \\ \mathcal{X}^{(3)} = \mathcal{X}^{(2)}(t) + \underline{\mathcal{X}_{32}(t)} \\ \mathcal{X}^{(4)} = \mathcal{X}^{(3)}(t) + \underline{\mathcal{X}_{43}(t)} \end{cases}$$

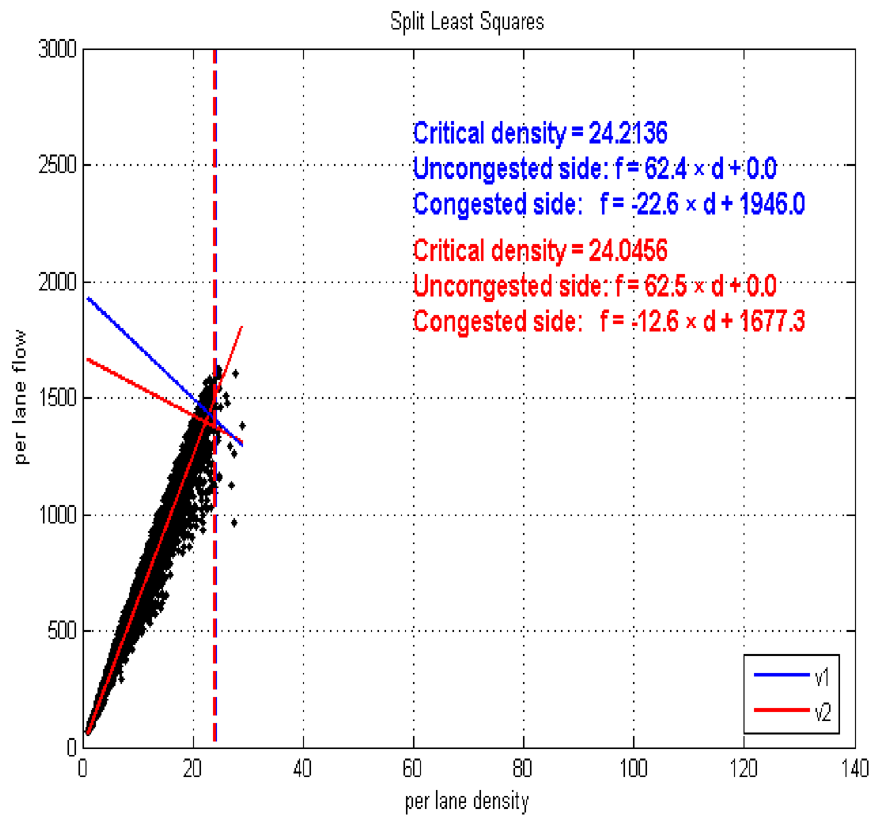
*small*

Network of team members

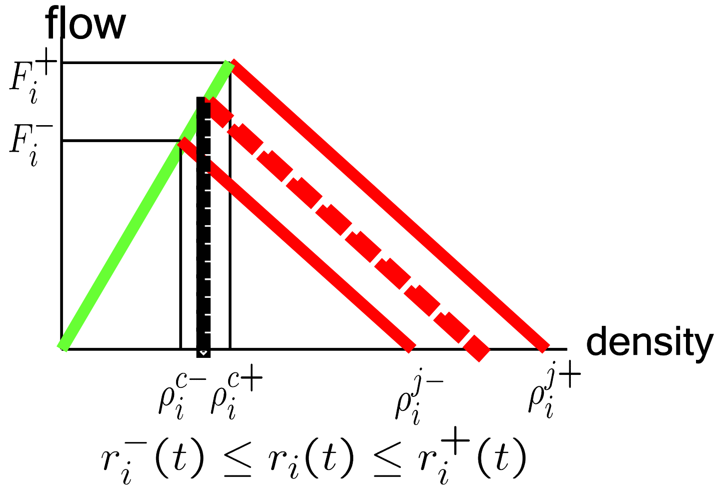


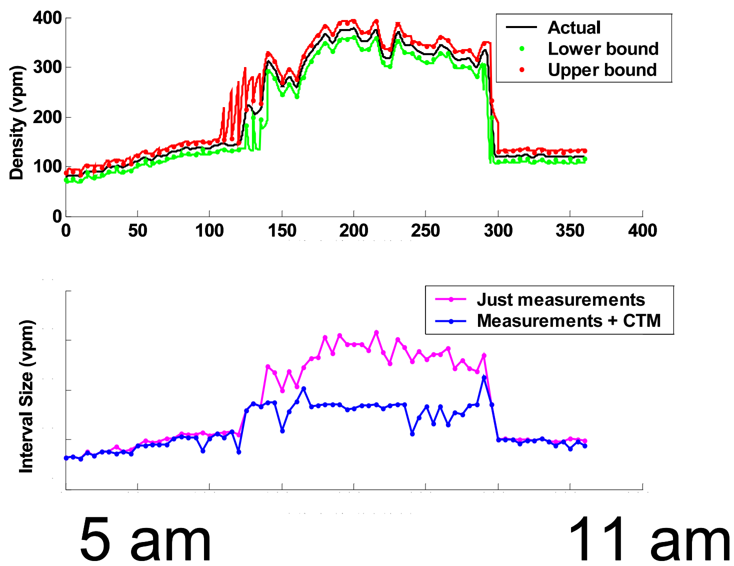

Split Least Squares
Critical density = 34.4888
Uncongested side: f = 56.3 × d + 0.0
Congested side: f = -6.9 × d + 1529.1
Critical density = 33.6969
Uncongested side: f = 56.6 × d + 0.0
Congested side: f = -6.9 × d + 1531.8
per lane flow
per lane density
v1
v2
0
500
1000
1500
2000
2500
3000
20
40
60
80
100
120
140

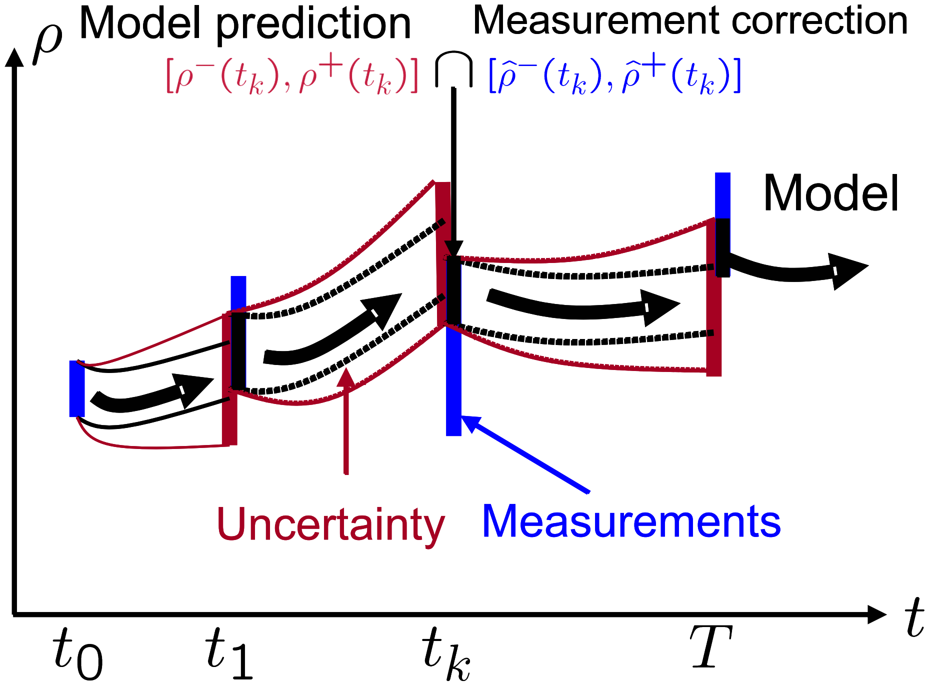
(a)

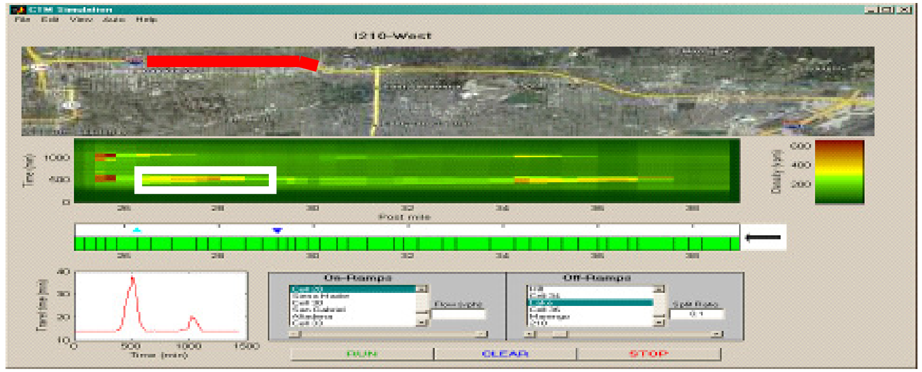

Split Least Squares
Critical density = 24.2136
Uncongested side: f = 62.4 × d + 0.0
Congested side: f = -22.6 × d + 1946.0
Critical density = 24.0456
Uncongested side: f = 62.5 × d + 0.0
Congested side: f = -12.6 × d + 1677.3
per lane flow
per lane density
v1
v2
0
500
1000
1500
2000
2500
3000
20
40
60
80
100
120
140

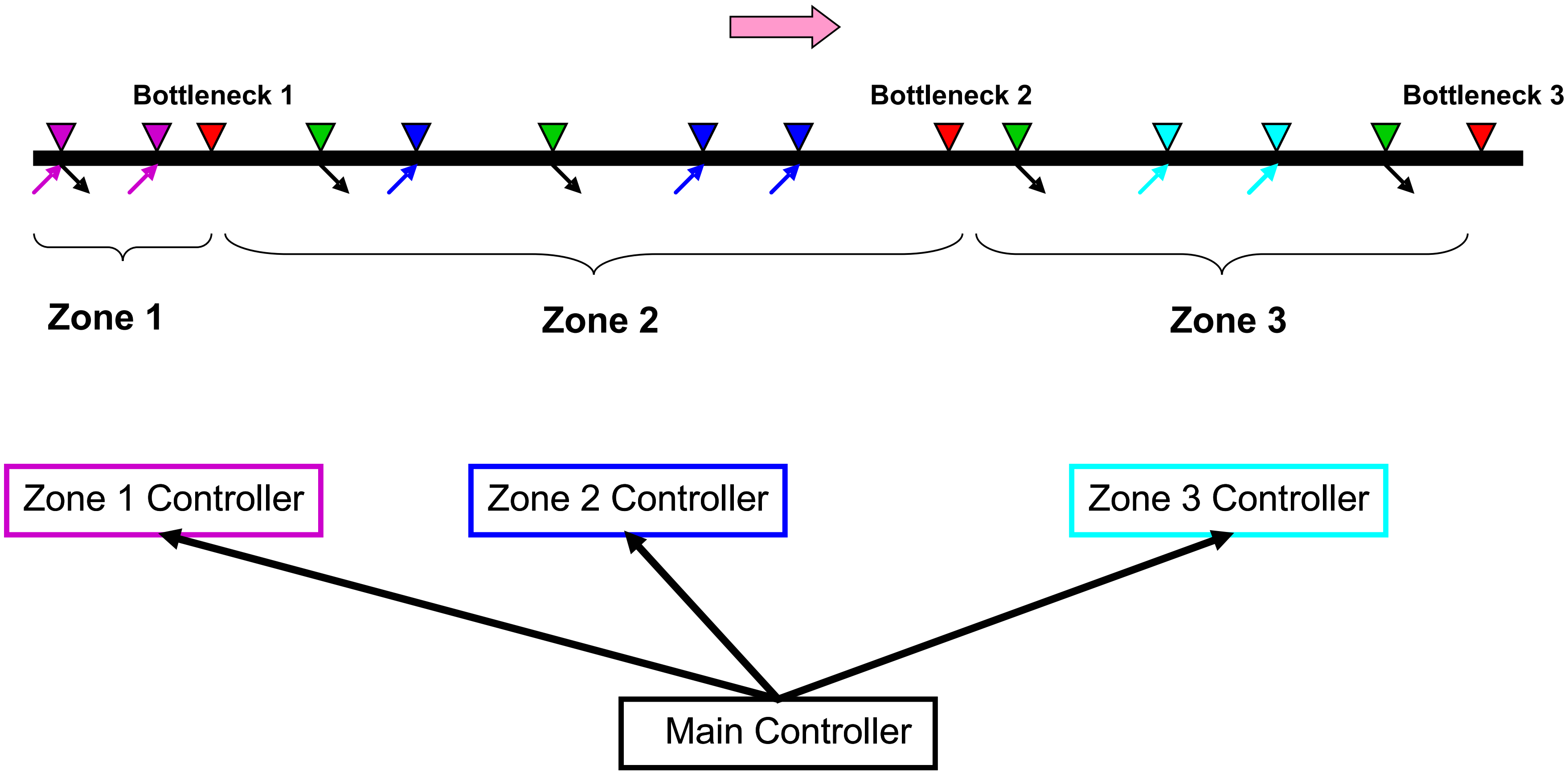
(b)


flow
$F_i^+$
$F_i^-$
density
$\rho_i^{c-}$ $\rho_i^{c+}$ $\rho_i^{j-}$ $\rho_i^{j+}$
$r_i^-(t) \leq r_i(t) \leq r_i^+(t)$


Density (vpm)
Actual
Lower bound
Upper bound
Interval Size (vpm)
Just measurements
Measurements + CTM
5 am
11 am


Model prediction
Measurement correction
$[\rho^-(t_k), \rho^+(t_k)] \bigcap [\hat{\rho}^-(t_k), \hat{\rho}^+(t_k)]$
Model
Uncertainty
Measurements
$\rho$
$t$
$t_0$ $t_1$ $t_k$ $T$




Bottleneck 1
Bottleneck 2
Bottleneck 3
Zone 1
Zone 2
Zone 3
Zone 1 Controller
Zone 2 Controller
Zone 3 Controller
Main Controller

# References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