Fast Controls and Their Calculation

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***Abstract*— New technologies, such as control in quantum systems, may require that the control would act on a very small time horizon. Another requirement is that the control should be designed in a closed-loop form. A possible response to this demand is the use of *fast controls* [1]. They are introduced as bounded approximations of generalized impulse controls (belonging to the class of higher-order distributions).**

I. Ideal Zero-Time Controls

Consider a linear differential equation in distributions:

$$\dot{x}(t) = A(t)x(t) + B(t)u + f^{(\alpha)} - f^{(\beta)},\\ x \in \mathbb{R}^n, \quad t \in [\alpha, \beta],$$

with k times continuously differentiable matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$. The terms $f^{(\alpha)}$ and $f^{(\beta)}$ are two distributions [2], [3] from $D^*_{k,n}[\alpha, \beta]$ concentrated in points t_α and t_β respectively.

Recall that the space $D^*_{k,m}$ consists of k times differentiable functions $\varphi(t) : [\alpha, \beta] \to \mathbb{R}^m$ with support set contained in $[\alpha, \beta]$, endowed with a norm

$$\mathscr{G}[\varphi] = \max_{t\in[\alpha,\beta]} \gamma[\gamma_0(\varphi(t)), \ldots, \gamma_k(\varphi^{(k)}(t))].$$

Here γ_k, γ are some finite-dimensional norms in vector spaces $\mathbb{R}^m$ and $\mathbb{R}^{k+1}$ respectively. The norm $\mathscr{G}[\varphi]$ defines a conjugate norm $\mathscr{G}^*[u]$ in the space $D^*_{k,m}[\alpha, \beta]$.

The generalized control $u(t)$ is chosen from the space $D^*_{k,m}[\alpha, \beta]$ and must ensure the existence of distribution $x(t) \in D^*_{k-1,n}[\alpha, \beta]$ with support on $[t_\alpha, t_\beta]$. This control may be represented in the form [4]

$$u(t) = \sum\nolimits_{j=0}^{k} \frac{d^{j+1}U_j(t)}{dt^{j+1}}, \quad U_j \in BV[t_\alpha, t_\beta].$$

Here $BV[t_\alpha, t_\beta]$ is a space of functions $U(t)$ of bounded variation.

In [5], [6] it was shown that a completely controllable linear system may be steered from one state to another in fixed time by an ordinary impulse control ($k = 0$)

$$u(t) = \sum\nolimits_{i=1}^{N} u_i \delta(t - \tau_i),$$

where the number of impulses $N \le n$. Here the points τ_i are time moments when the impulses are applied to the system.

For $k > 0$ the controls include, apart from delta-functions, higher-order derivatives of delta-functions, which extends capabilities of the control. In particular, for $m \ge n - 1$ a completely controllable linear system may be steered from one state to another by the control of type

$$u(t) = \sum\nolimits_{j=0}^{m} u_j \delta^{(j)}(t - \tau). \tag{1}$$

in zero time.

The problem of feedback control in the class of distributions is solved using Hamilton–Jacobi–Bellman type variational inequalities [1].

II. Fast Controls

Control (1) is an “ideal” one. Bounded functions approximating (1) are known as *fast controls*, since they are physically realizable and may steer a system to a given state in arbitrary small time [1]. Such controls may be found, for example, in the following form:

$$u_\Delta(t) = \sum\nolimits_{j=0}^{m} u_j \Delta^{(j)}_{h_j}(t - \tau), \tag{2}$$

where $\Delta^{(j)}_h(t)$ approximate the derivatives of delta-function:

$$\Delta^{(0)}_h(t) = h^{-1}\mathbf{1}_{[0,h]}(t),\\ \Delta^{(j)}_h(t) = h^{-1}\left(\Delta^{(j-1)}_h(t) - \Delta^{(j-1)}_h(t-h)\right).$$

Here arises the problem of how to choose the parameters of control (2) — the coefficients h_j and vectors u_j. These parameters should be chosen due to physical requirements on the realizations of control.

III. Discontinuous Approximations

First, we consider fast controls with various restrictions:

1) bounded time of control:
$$\max_j\{(j+1)h_j\} \le H;$$
2) hard bounds on control:
$$\|u_\Delta(t)\| \le \mu;$$
3) separate hard bounds on approximations of generalized functions of all orders included in the control:
$$\|u_{\Delta,j}(t)\| \le \mu_j,\\ u_{\Delta,j}(t) = u_j\Delta^{(j)}_{h_j}(t - \tau).$$

The indicated restrictions lead to the problems of moments of similar type. For example, for an approximation of the derivative $\delta^{(n)}(t)$ such a moment problem has the following solution:

$$\Delta^n_h(t) = \tfrac{1}{4}(-1)^n n! \left(\tfrac{2}{h}\right)^{(n+1)} \operatorname{sign} U_n(t/h), \tag{3}$$

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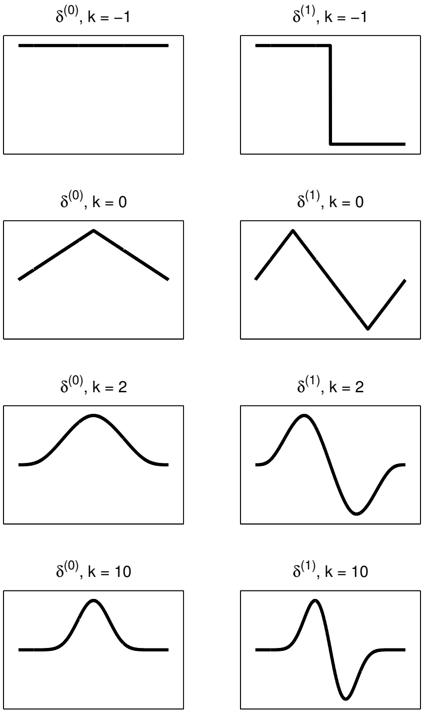

Fig. 1. Approximations of $\delta^{(n)}(t)$ (Part I)

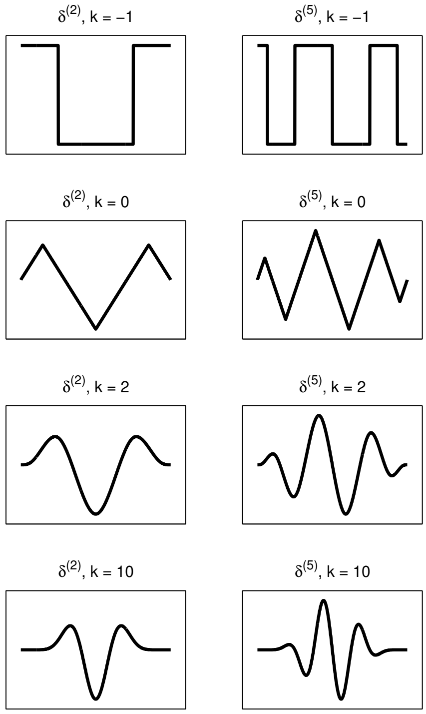

Fig. 2. Approximations of $\delta^{(n)}(t)$ (Part II)

where $U_n(t)$ is the Chebyshev polynomial of the second kind:

$$U_n(t) = \cos(n \arccos t).$$

The approximation (3) is piecewise constant (and hence discontinuous), equal to $\pm\frac{1}{4}n!\left(\frac{2}{h}\right)^{(n+1)}$ between Chebyshev points $t_k = h\cos\frac{\pi j}{n+1}$, $j = 0, \ldots, n+1$.

A. Haar Series

Another promising approach is to represent fast controls in terms of Haar series [7], since the basic element (wavelet) of this orthogonal system of functions is essentially an approximation of $\delta'(t)$:

$$\chi_n^{(k)}(t) = \begin{cases} \sqrt{2^n}, & t \in \left(\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right); \\ -\sqrt{2^n}, & t \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right); \\ 0, & \text{otherwise}; \end{cases}$$
$$n = 0, 1, \ldots, \quad k = 1, \ldots, 2^n.$$

IV. Smooth Approximations

After that, we consider continuous or smooth approximations. To do this, we impose the bound on k-th derivative of the approximation $\|u_\Delta^{(k)}(t)\| \leq \mu$. It turns out that a k times smooth approximation of $\delta^{(n)}(t)$, $\Delta_{h,k}^n(t)$, is a normalized $(k+1)$-fold integral of $\Delta_h^{n+k+1}(t)$. Here $k = -1$ corresponds to discontinuous approximations $\Delta_h^n(t)$, and $k = 0$ leads to continuous (but not smooth) approximations.

Approximations $\Delta_{h,k}^n(t)$ are piecewise polynomials of order k, with $k-1$ derivatives continuous at the junction points. The coefficients of these polynomials may be calculated recursively by explicit formulae.

V. Examples

In Figs. 1, 2 we present our approximations of $\delta(t)$ and its derivatives, with various degrees of smoothness.

References

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