Fast Controls and Their Calculation

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Introduction

Impulse controls:

- instantaneous control actions ("hits")
- trajectories with discontinuities, jumps, resets, etc.

Mechanical systems:

- Using ordinary δ-functions: gives jump in velocity
- Using higher derivatives of δ-functions: gives reset of all coordinates.

Physically realizable approximations
of zero-time controls

$$\Downarrow$$

Fast Controls

Introduction

How to calculate Fast Controls?

- time bound
- intensity bound

Impulse Control Problem

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$t \in [t_0, t_1]$ — a *fixed* time interval

Problem (1)

$$\textit{Minimize } J(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1 + 0))$$

over $U(\cdot) \in BV[t_0, t_1]$*, where* $x(t)$ *corresponds to control input*

$$u(t) = \frac{dU}{dt}$$

and starts from $x(t_0 - 0) = x_0$.

Impulse Control Problem

The classical result
(N. N. Krasovski [1957], L. W. Neustadt [1964]):

$$u(t) = \sum_{i=1}^{n} h_i \delta(t - \tau_i)$$

Important particular case: $\varphi(x) = \mathscr{I}\left(x \mid \{x_1\}\right)$
— steer from x_0 to x_1 on $[t_0, t_1]$.

$$\mathscr{I}\left(x \mid A\right) = \begin{cases} 0, & x \in A; \\ +\infty, & x \notin A. \end{cases}$$

Dynamic Programming

Definition

The Value Function is the minimum of $J(U(\cdot))$ for a *fixed* initial state $x(t_0 - 0) = x_0$:

$$V(t_0, x_0) = V(t_0, x_0; t_1, \varphi(\cdot)).$$

How to calculate the Value Function?

- Integrate the HJB equation.
- Explicit representation (from convex analysis).

Dynamic Programming

The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the
principle of optimality

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))), \quad \tau \in [t_0, t_1]$$

Variational inequality of Hamilton–Jacobi–Bellman type:

$$\min\left\{H_1(t, x, V_t, V_x),\ H_2(t, x, V_t, V_x)\right\} = 0,$$

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)).$$

$$H_1 = V_t + \langle V_x, A(t)x\rangle, \quad H_2 = \min_{u \in S_1} \langle V_x, B(t)u\rangle + 1 = -\left\|B^T(t) V_x\right\| + 1.$$

Explicit Representation for V

Denote

- $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.
- $\|p\|_{[t_0,t_1]} = \left\| B^T(\cdot)X^T(t_1,\cdot)p \right\|_{C[t_0,t_1]}$,

The Value Function

$$V(t_0,x_0) = \inf_{x_1\in\mathbb{R}^n}\left\{\varphi(x_1) + \sup_{p\in\mathbb{R}^n}\frac{\langle p, x_1 - X(t_1,t_0)x_0\rangle}{\|p\|_{[t_0,t_1]}}\right\}.$$

The Conjugate Function

$$V^*(t_0,p) = \varphi^*(X^T(t_0,t_1)p) + \mathscr{I}\left(X^T(t_0,t_1)p \;\middle|\; \mathscr{B}_{\|\cdot\|_{[t_0,t_1]}}\right)$$

Explicit Representation: $t_1 - t_0 = 0$

With $t_1 = t_0$, there is only one instant available for control.

This instant may be used in an efficient way.

The Value Function

$$V(t_1, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \|x_1 - x_0\|^*_{[t_0, t_1]} \right\} \leqslant \varphi(x_0).$$

The Conjugate Function

$$V^*(t_1, p) = \varphi^*(p) + \mathscr{I}(p \mid \{p \mid \|B^T(t_1)p\| \leqslant 1\})$$

Generalized Control Problem

Problem (2)

$$\textit{Minimize } J(u) = \rho^*[u] + \varphi(x(t_1 + 0))$$

over distributions $u \in D_k^*[\alpha, \beta]$, $(\alpha, \beta) \supseteq [t_0, t_1]$,
where $x(t)$ *is the trajectory under control* u
and initial state $x(t_0 - 0) = x_0$.

$\rho^*[u]$ is the conjugate norm for the norm ρ on $C^k[\alpha, \beta]$:

$$\rho[\psi] = \max_{t \in [\alpha, \beta]} \sqrt{\|\psi(t)\|^2 + \|\psi'(t)\|^2 + \cdots + \left\|\psi^{(k)}(t)\right\|^2}.$$

$$u(t) = \sum_{i=1}^{n} h_i^{(0)} \delta(t - \tau_i) + h_i^{(1)} \delta'(t - \tau_i) + \cdots + h_i^{(k)} \delta^{(k)}(t - \tau_i).$$

Reduction to "Ordinary" Impulse Controls

How to deal with higher derivatives $\delta^{(j)}(t)$?

Reduce to a problem with ordinary δ-functions,

but for a more complex system.

General form of distributions $u \in D_k^*$:

$$\boxed{u = \frac{dU_0}{dt} + \frac{d^2U_1}{dt^2} + \cdots + \frac{d^kU_k}{dt^k}} \quad U_j \in BV$$

Problem 2 reduces to a particular case of Problem 1 for the system

$$\dot{x} = A(t)x + \mathscr{B}(t)u, \quad \mathscr{B}(t) = \begin{pmatrix} L_0(t) & L_1(t) & \cdots & L_k(t) \end{pmatrix}$$

and control $u = \dfrac{dU}{dt}$, $U(t) = \begin{pmatrix} U_0(t) \\ \vdots \\ U_k(t) \end{pmatrix}$,

with $L_0(t) = B(t)$, $L_j(t) = A(t)L_{j-1}(t) - L'_{j-1}(t)$.

Reduction to "Ordinary" Impulse Controls

$$\mathscr{B}(t) = \begin{pmatrix} L_0(t) & L_1(t) & \cdots & L_k(t) \end{pmatrix}$$

For example:

- $A = 0$:

$$\mathscr{B}(t) = \begin{pmatrix} B(t) & -B'(t) & B''(t) & \cdots & (-1)^k B^{(k)}(t) \end{pmatrix}$$

- $A, B = \mathrm{const}$:

$$\mathscr{B}(t) = \begin{pmatrix} B & AB & A^2B & \cdots & A^kB \end{pmatrix}$$

Fast Controls

If rank $\mathscr{B} = n$, the system may be steered from x_0 to x_1 in zero time by the control

$$u(t) = h^{(0)}\delta(t - t_0) + h^{(1)}\delta'(t - t_0) + \cdots + h^{(k)}\delta^{(k)}(t - t_0).$$

i.e.

$$x_1 - x_0 = L_0(t_0)h^{(0)} + L_1(t_0)h^{(1)} + \cdots + L_k(t_0)h^{(k)}.$$

Approximations of zero-time controls are
Fast Controls.

Control in arbitrary small “fast” time (“nano-time”).

Fast Controls — 1

Finite-difference approximation of derivatives of the delta-function:

$$\Delta_\sigma^0(t) = \frac{1}{\sigma}\mathbf{1}_{[0,\sigma]}(t), \quad \Delta_\sigma^j(t) = \frac{1}{\sigma}(\Delta_\sigma^{j-1}(t) - \Delta_\sigma^{j-1}(t-\sigma))$$

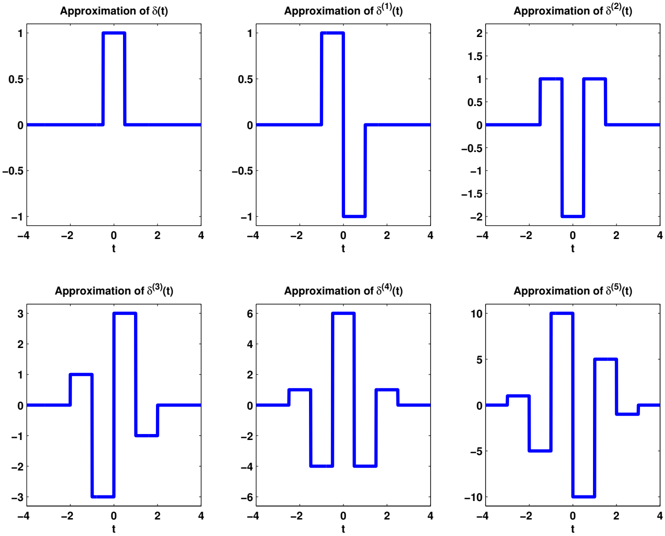
Zero-time control

$$u(t) = h^{(0)}\delta(t-t_0) + h^{(1)}\delta'(t-t_0) + \cdots + h^{(k)}\delta^{(k)}(t-t_0)$$

is replaced with a **fast control**

$$u_\sigma(t) = h^{(0)}\Delta_\sigma^0(t-t_0) + h^{(1)}\Delta_\sigma^1(t-t_0) + \cdots + h^{(k)}\Delta_\sigma^k(t-t_0)$$

Fast Controls — 1



Fast Controls — 1

The problem with fast controls
is reduced to the impulse control problem

$$\dot{x} = A(t)x + \mathscr{M}_\sigma(t)u, \quad \mathscr{M}_\sigma(t) = \begin{pmatrix} M_\sigma^{(0)}(t) & M_\sigma^{(1)}(t) & \cdots & M_\sigma^{(k)}(t) \end{pmatrix}$$

where

$$M_\sigma^{(j)}(t) = \int_t^{t+k\sigma} X(t+k\sigma,\tau)B(\tau)\Delta_\sigma^j(\tau - t)d\tau$$

We have $\mathscr{M}_\sigma(t) \to \mathscr{B}(t)$ as $\sigma \to 0$.

Fast Controls — 2

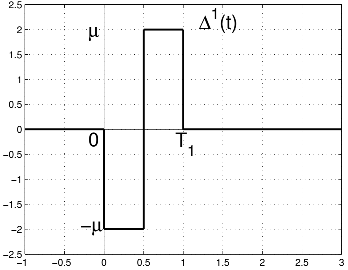
- Intensity bound $|\Delta^k(t)| \leqslant \mu$
  for each of the derivatives

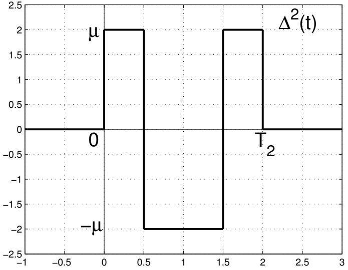
$$\Delta_\mu^0(t) = \mu \mathbf{1}_{[0,h_0]}(t), \quad h_0 = \mu^{-1}.$$

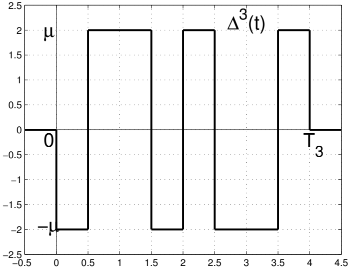
$$\Delta_\mu^k(t) = \Delta_\mu^{k-1}\left(\left(\tfrac{1}{2}t\right)^{\frac{k+1}{k}} - h_{k-1}\right) - \Delta_\mu^{(k-1)}\left(\left(\tfrac{1}{2}t\right)^{\frac{k+1}{k}}\right)$$

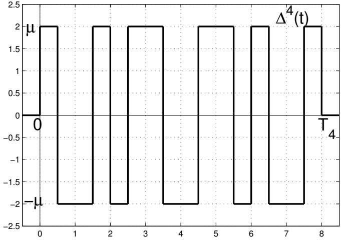
$$h_k = 2(h_{k-1})^{\frac{k}{k+1}}$$

Fast Controls — 2









- Intensity bound $|\Delta^k(t)| \leqslant \mu$
- Minimum time h_k

$$\Updownarrow$$

- Time bound h_k
- Minimum intensity $|\Delta^k(t)|$

Reduces to a moments problem:

$$\mu \to \inf,$$
$$|\Delta_h^n(t)| \leqslant \mu, \quad t \in [-h, h],$$

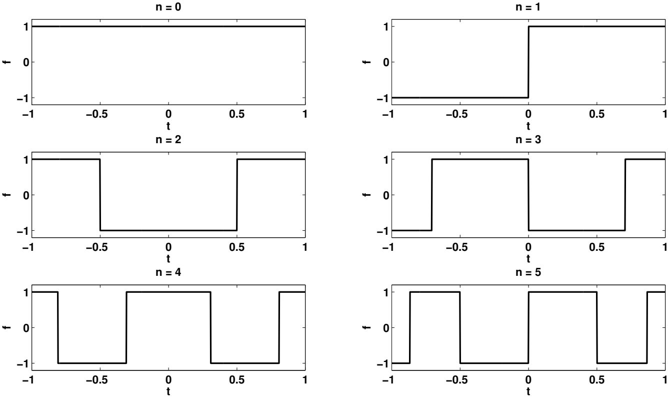
$$\int_{-h}^{h} \Delta_h^n(t) t^k dt = 0, \quad k = 0, \ldots, n-1,$$
$$\int_{-h}^{h} \Delta_h^n(t) t^n dt = (-1)^n n!$$

Solution:

$$\Delta_h^n(t) = \frac{1}{8}(-1)^n n! \left(\frac{2}{h}\right)^{(n+1)} \operatorname{sign} U_n(ht).$$

$U_n(t)$ — Chebyshev polynomial of the second kind.

Fast Controls — 3



Fast Controls — 3

Switching times:

$$t_k = h \cos \frac{\pi k}{n+1}, \quad k = 1, \dots, n.$$

Estimate for polynomials of higher degrees:

$$\left| \int_{-h}^{h} \Delta_h^n(t) t^k dt \right| \leqslant 2^{n-1} \frac{n!}{k+1} h^{k-n}, k = n+1, n+2, \dots .$$

Fast Controls — 4

- Bounded k-th derivative: $|(\Delta_h^n(t))^{(k)}| \leqslant \mu$
- Minimum time h_k

$$\Downarrow$$

- Continuous or Smooth approximations

Fast Controls — 4

Moments problem:

$$\mu \to \inf,$$

$$\Delta^n_{h,k} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{k-1}} g(\tau_k) d\tau_k \, |g(t)| \leqslant \mu, \quad t \in [-h, h],$$

$$\int_{-h}^{h} \Delta^n_h(t) t^k dt = 0, \quad k = 0, \ldots, n-1,$$

$$\int_{-h}^{h} \Delta^n_h(t) t^n dt = (-1)^n n!$$

Moments problem for $g(\tau)$:

$$\mu \to \inf,$$
$$|g(t)| \leqslant \mu, \quad t \in [-h, h],$$

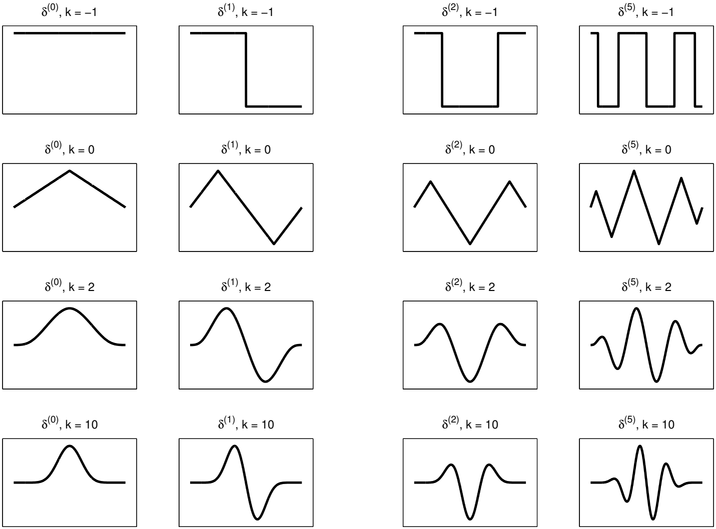
$$\int_{-h}^{h} \Delta_h^n(t) t^k dt = 0, \quad k = 0, \ldots, n+k-1,$$
$$\int_{-h}^{h} \Delta_h^n(t) t^{n+k} dt = c$$

Fast Controls — 4

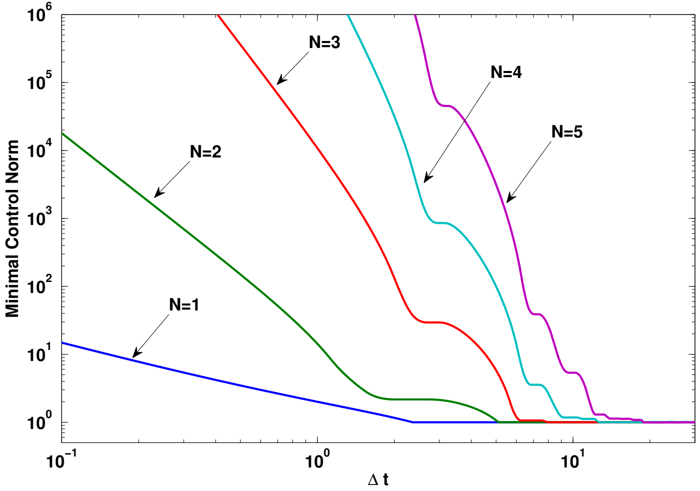
Solution:

$$\Delta_{h,k}^{n} = C\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{k-1}} \Delta_h^{n+k}(\tau_k)d\tau_k$$

Fast Controls — 4



Fast Controls: Growth Rate



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