The Control of Linear Systems under Feedback Delays

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Introduction

The emphasis in this paper is on

- Feedback delays
- Measurement feedback
- Set-membership noise
- Effect of delay
- Systems of high dimensions

The **solution** is based on

- Hamiltonian techniques
- Convex analysis
- Ellipsoidal calculus

Numerical modelling

- Effect of delay
- Oscillating systems
- High dimensions (here up to 20)

Introduction

Problems:

1. Feedback delay, no noise
2. Noisy measurement feedback, no delay
3. Noisy measurement feedback + delay

Problem 1 - Feedback delay, no noise

System:

$$\dot{x}(t) = A(t)x(t) + B(t)u$$

on **fixed** time interval $t \in [t_0, t_1]$.

- Hard bound: $u[t] \in \mathscr{P}(t) \in \operatorname{conv} \mathbb{R}^n$
- Feedback control: $u = u[t] = U(t, x(t-h))$.
- The controller is allowed to be with memory.
- Target control: $x(t_1) \in \mathscr{M}$.

Problem 1 - Feedback delay, no noise

Solution: reduce to system without delay.

- For $t \in [t_0, t_0 + h)$: set $u = 0$.
- For $t \geqslant t_0 + h$: consider system

$$\dot{z}(t) = A(t)z(t) + B(t)u[t], \quad z(t_0 + h) = X(t_0 + h, t_0)x(t_0).$$

(here $\partial X(t, \tau)/\partial t = A(t)X(t, \tau)$, $X(\tau, \tau) = I$).

State $z(t)$ is available **without delay** $\Rightarrow$ construct feedback control $u = U(t, z)$.

Reset at time τ:

$$z(\tau) := X(\tau, \tau - h)x(\tau - h) + z_\tau(\tau).$$

$$\dot{z}_\tau(t) = A(t)z_\tau(t) + B(t)u[t], \quad z_\tau(\tau - h) = 0, \quad t \in [\tau - h, \tau]$$

Problem 2 - Measurement feedback, no delay

Measurement equation:

$$y(t) = H(t)x(t) + \xi(t)$$

- Hard bound: $\xi[t] \in \mathscr{Q}(t) \in \operatorname{conv} \mathbb{R}^n$
- Feedback control: $u = u[t] = U(t, y(t))$ with memory.

Problem 2 - Measurement feedback, no delay

Solution:

- Information set $\mathscr{X}[\tau]$ (guaranteed state estimation)

$$\lim_{\sigma\to 0+0} \sigma^{-1} h(\mathscr{X}(t+\sigma), (\mathscr{X}(t)+\sigma B(t)u^*[t])\cap(y^*(t)-\mathscr{Q}(t))) = 0.$$

- State $\{\tau, \mathscr{X}[\tau]\} \Rightarrow$ **infinite-dimension** problem (metric space of convex compacts).
- But: it reduces to a **finite-dimension** problem through techniques of convex analysis (see paper for details).

Problem 3 - Measurement feedback + delay

Measurement equation:

$$y(t-h) = H(t-h)x(t-h) + \xi(t-h), \quad \xi(t-h) \in \mathscr{Q}(t-h).$$

- Time delay h.
- Feedback control: $u = u[t] = U(t, y(t-h))$ with memory.

Solution: combine techniques for Problems 1 and 2

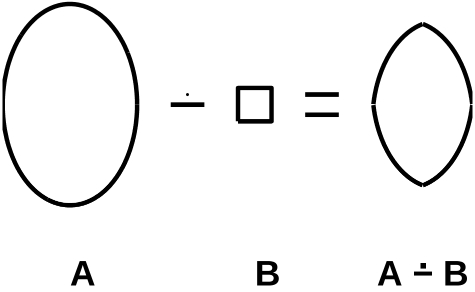
- State: $\{t, \mathscr{X}(t-h), u[t-h, t]\}$
- Notation: $\mathbf{T}_u u[\tau, t] = \int_\tau^t X(t, \vartheta) B(\vartheta) u(\vartheta) d\vartheta.$
- Guaranteed estimate of the current position $x(t)$:

$$\mathscr{X}^*[t] = X(t, t-h)\mathscr{X}(t-h) + \mathbf{T}_u u[t-h, t].$$

Problem 3 - Measurement feedback + delay

Notation: Geometric (Minkowski) difference:

$$A \dot{-} B = \{x \in \mathbb{R}^n \mid x + B \subseteq A\}$$



Problem 3 - Measurement feedback + delay

Solution (cont.):

- Estimate of the value function:

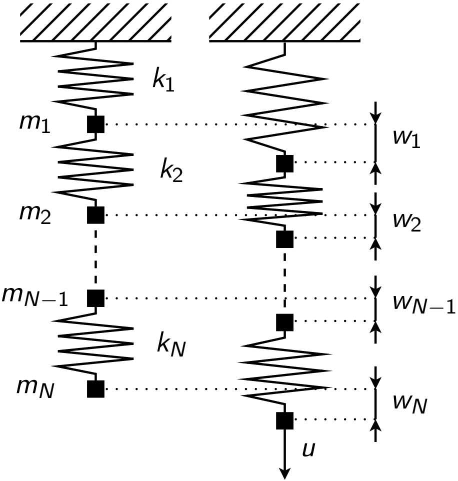
$$\mathscr{V}(t, \mathscr{X}(t-h), u[t-h,t]) \leqslant$$

$$\leqslant d(X(t_1,t)\mathbf{T}_u u[t-h,t],$$

$$\mathscr{M} \dot{-} X(t_1, t-h)\mathscr{X}(t-h) - \mathbf{T}_u \mathscr{P}[t,t_1])$$

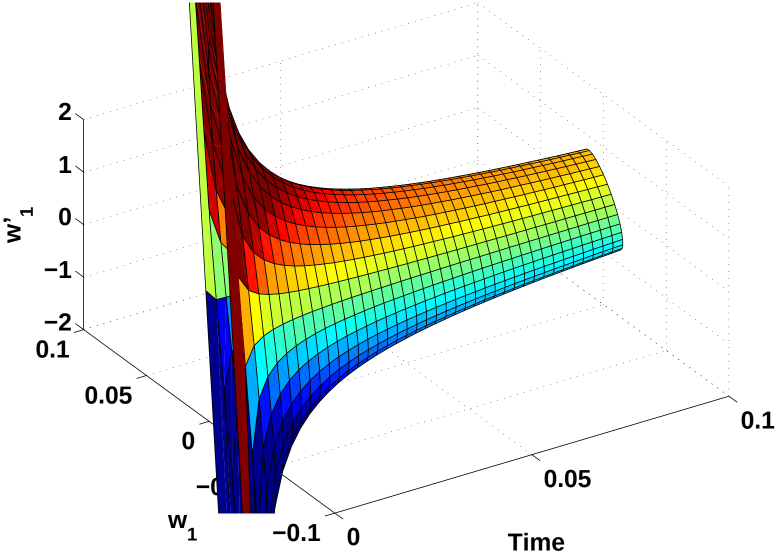
- It is equal to value function of a linear-convex problem with:
  - Target set $\mathscr{M} \dot{-} X(t_1, t-h)\mathscr{X}(t-h)$
  - Initial position $x(t) = \mathbf{T}_u u[t-h,t]$
  - No delay
  - No noise
- Apply **Ellipsoidal Calculus** to this problem.

Examples

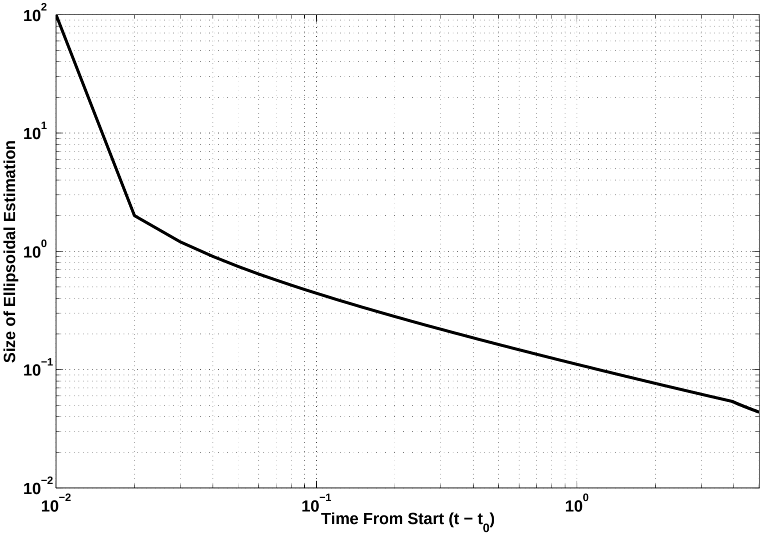


- Only positions w_i measured
- Control applied to lower weight
- Completely controllable
- Completely observable

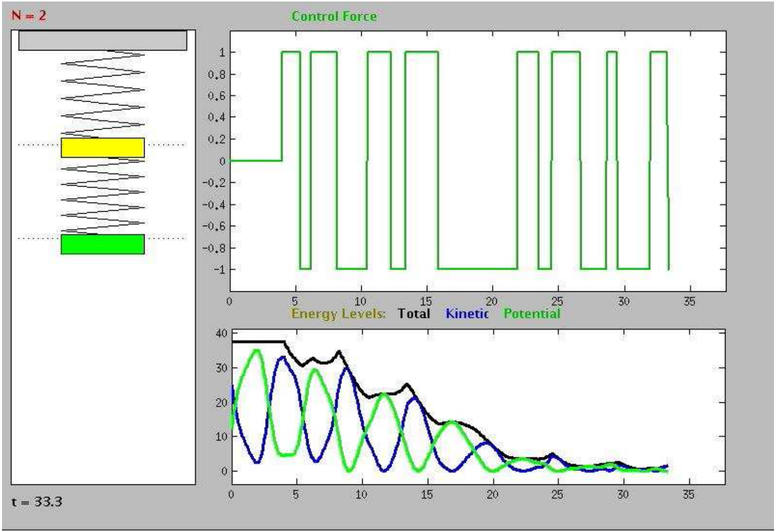
$N = 1$: Information Tube



Size of the Information Tube

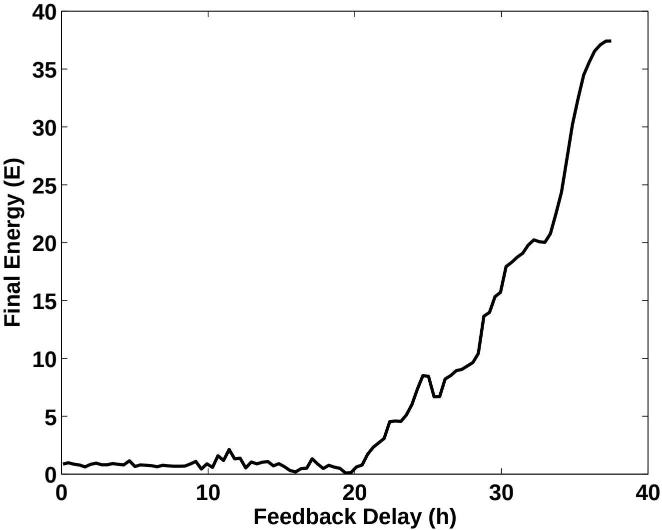


$N = 2$

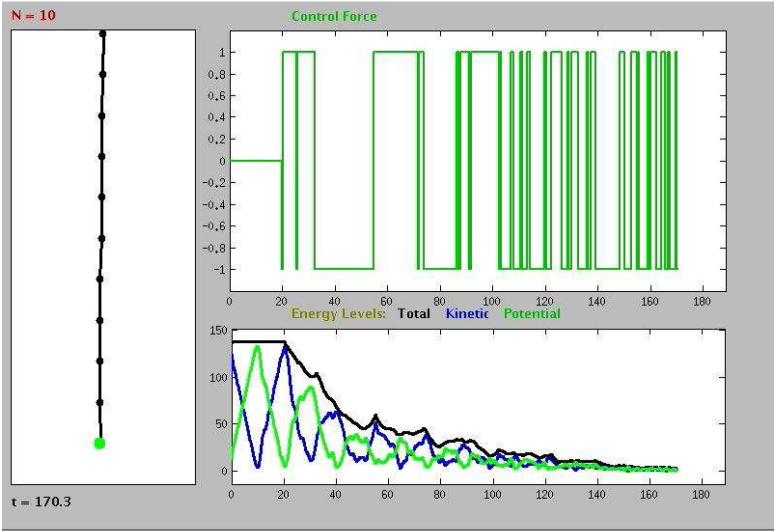


N = 2
Control Force
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
35
Energy Levels: Total Kinetic Potential
40
30
20
10
0
0
5
10
15
20
25
30
35
t = 33.3

$N = 2$: Effect of Delay



$N = 10$



N = 10
Control Force
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
20
40
60
80
100
120
140
160
180
Energy Levels: Total Kinetic Potential
150
100
50
0
0
20
40
60
80
100
120
140
160
180
t = 170.3

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