Reachability Approaches and Ellipsoidal Techniques for Closed-Loop Control of Oscillating Systems under Uncertainty

A. N. Daryin, A. B. Kurzhanski and I. V. Vostrikov

***Abstract*— This paper indicates effective solution schemes for problems of closed-loop control for oscillating systems of high dimensions subjected to unknown but bounded disturbances. These schemes are based on using internal ellipsoidal approximations of weakly invariant solvability sets and produce effective numerical algorithms. The text is accompanied by computer animation and numerical simulations based on ellipsoidal tools for calculating reach sets.**

I. Introduction

Within the range of important applied control problems under present investigation is the one of feedback control of oscillating systems of high dimensions. The given paper indicates effective schemes for solving such problems for systems subjected to unknown but bounded disturbances including those that may also be in resonance with system. The selected approach is based on constructing weakly invariant sets (similar to "Krasovski bridges") which are then further used to design the specific solution strategies and also to investigate the damping possibilities of the controls under various types of loads and disturbances. The text is accompanied by computer animation and numerical simulations based on ellipsoidal tools for calculating reach sets. This allows to tackle problems of realistically high dimensions and solve them in limited time.

II. The Problem

The problem we consider in this paper is to design a feedback synthesizing strategy to damp the oscillations of a suspended chain of a finite number of loaded springs by applying a bounded control force to the lower end of the chain (Fig. 1). The chain must be led to an equilibrium in given finite time, so that this is *not* a problem of asymptotic stabilization.

Apart from the springs, the chain also includes given loads attached in between the springs. We assume that the masses of springs are negligibly small compared to those of the loads. The upper end of the chain is rigidly attached to a fixed suspension. Then the oscillations of the chain could be described by the following system of second-order ODEs:

$$\begin{cases} m_1\ddot{w}_1 = k_2(w_2 - w_1) - k_1 w_1, \\ m_i\ddot{w}_i = k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}), \\ m_N\ddot{w}_N = -k_N(w_N - w_{N-1}) + u(t), \end{cases} \tag{1}$$

when $t > t_0$. Here N is the number of springs which are numbered from top to bottom. The loads are numbered similarly, so that the i-th load is attached to the lower end of the i-th spring. w_i is the displacement of the i-th load from the equilibrium, m_i is the mass of the i-th load, k_i is the stiffness coefficient of the i-th spring. The gravity force enters (1) implicitly through determining the lengths of the springs at the equilibrium.

The initial state of the chain at time t_0 is given by the displacements w_i^0 and the velocities of the loads $\dot{w}_i^0$:

$$\begin{cases} w_i(t_0) = w_i^0, \\ \dot{w}_i(t_0) = \dot{w}_i^0. \end{cases}$$

The control $u(t)$ is a scalar taking values from a given interval $\mathscr{P} = [u_{\min}, u_{\max}]$ which defines the *hard bound*, or *geometric constraint* on the control. Thus the absolute value of the control may be bounded by value $\mu > 0$: $\mathscr{P} = [-\mu, \mu]$, or the control may be applied just in one direction (only down: $\mathscr{P} = [0, \mu]$, or only up: $\mathscr{P} = [-\mu, 0]$).

The equations (1) may be interpreted as a spatial discretization of a one-dimensional wave equation for a string with fixed left end and a control force applied to the free

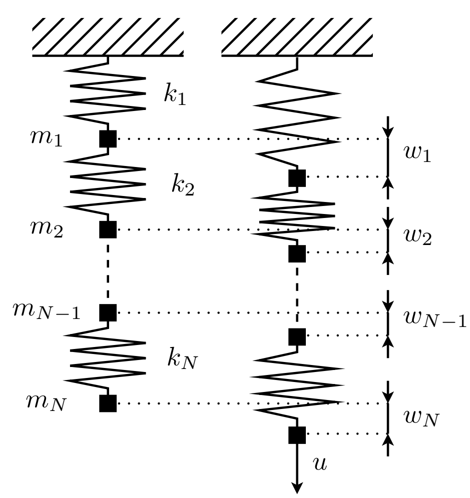

Fig. 1. The chain of springs to be controlled in the equilibrium state (left) and in an arbitrary state (right)

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right end:

$$\rho(\xi)w_{tt}(t,\xi) = [E(\xi)w_\xi(t,\xi)]_\xi, \quad t > t_0, \quad 0 < \xi < L;$$
$$w(t,0) = 0, \quad w_\xi(t,L) = E^{-1}(L)u(t), \quad t \geqslant t_0;$$
$$w(t_0,\xi) = w^0(\xi), \quad w_t(t_0,\xi) = \dot{w}^0(\xi), \quad 0 \leqslant \xi \leqslant L.$$

(Here $w(t,\xi)$ is the displacement of point ξ at time t; each point ξ of the string is characterized by Young modulus $E(\xi)$ and mass density $\rho(\xi)$.) Therefore, the presented approach may be also useful for investigating problems of boundary control for the wave equation.

The goal of the control is to steer the system to the equilibrium in given finite time [1]. In other words, one has to specify a closed-loop feedback control strategy $u(\cdot,\cdot)$, which steers the system (1) from any given initial state to the origin at given time t_1: $w(t_1) = 0$, $\dot{w}(t_2) = 0$.

In the absence of uncertainty the open-loop and feedback controls achieve the same result, i.e. the problem is solvable in $\mathscr{U}_{\mathrm{OL}}$ if and only if it is solvable in $\mathscr{U}_{\mathrm{CL}}$ (see [3]). However, if the disturbances are present in the system, a feedback control may yield a significantly better result by employing additional information which arrives during the control process.

Before providing an accurate definition of $\mathscr{U}_{\mathrm{CL}}$ and formulating the problem, we rewrite the original system (1) in the normalized matrix form. To do this, we introduce an extended state vector $x \in \mathbb{R}^n$, $n = 2N$, defined by $(x_1,\dots,x_N) = w$, $(x_{N+1},\dots,x_{2N}) = \dot{w}$. Then

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(t_0) = x^0 = \begin{pmatrix} w^0 \\ \dot{w}^0 \end{pmatrix}, \tag{2}$$
$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix}, \quad b = (0,\dots,0,m_N^{-1})^T,$$
$$M = \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_N \end{pmatrix},$$
$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{N-1} & k_{N-1}+k_N & -k_N \\ & & & -k_N & k_N \end{pmatrix}.$$

Note that the controllability property of the system depends on at which of the loads we apply the control force. For example, if $m_i \equiv m$, $k_i \equiv k$, and the control force is applied to n-th load, then the system is controllable if and only if $\gcd(n, 2N+1) = 1$. If the force is applied to the last N-th load, then the system is always controllable regardless of the values of masses and stiffness coefficients.

We now pass to a rigorous definition of the desired control strategy.

Definition 1: A closed-loop control is a multi-valued mapping $\mathscr{U}(t,x)$, upper semicontinuous in x, measurable in t, with non-empty convex compact values in $\mathscr{P}$. The class of closed-loop controls is denoted as $\mathscr{U}_{\mathrm{CL}}$.

Controls of the specified class ensure the existence and extendability of solutions to the differential inclusion [4]

$$\dot{x}(t) \in Ax(t) + b\mathscr{U}(t,x), \quad t \geqslant t_0 \tag{3}$$

arising by substituting the feedback control $\mathscr{U}(t,x)$ into the right-hand side of (2). Although the original system is linear in x, the closed-loop system (3) is not, due to the nonlinear term $\mathscr{U}(t,x)$. The effectiveness of using such a class of strategies is well known.

Problem 1: Find the *solvability domain* $\mathscr{W}[t_0] \subseteq \mathbb{R}^n$ and a feedback control $\mathscr{U}(t,x)$ such that all trajectories of (3) which originate in $\mathscr{W}[t_0]$ at time t_0 arrive at the equilibrium at time t_1 (that is, $x(t_1) = 0$).

The solvability domain $\mathscr{W}[t_0]$ is also known as the *backward reach set* (this is the set of all points reachable from the equilibrium in backward time).

This problem can be naturally extended to systems with uncertainty. Here we consider a problem of guaranteed control synthesis: the control has to damp the springs despite the unknown disturbances. We assume the admissible range of the disturbances to be known, thus following a set-membership description of these.

In particular, such approach allows taking into account the modeling errors. For example, if the upper end of the chain is not fixed rigidly enough, then some additional terms will arise in the first equation of (1). The physical parameters — masses and stiffness coefficients may be also known with certain errors in (1) which could also be taken to be bounded. One may similarly take into account the nonlinearities in describing the stiffness coefficients.

The system with disturbances is as follows

$$\begin{cases} m_1\ddot{w}_1 = k_2(w_2 - w_1) - k_1 w_1 + v_1, \\ m_i\ddot{w}_i = k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}) + v_i, \\ m_N\ddot{w}_N = -k_N(w_N - w_{N-1}) + u + v_N, \end{cases} \tag{4}$$

Here the disturbing force v_i is applied to the i-th load. As the control, the disturbance is also taken to be restricted by a hard bound $v(t) \in \mathscr{Q}$, where $\mathscr{Q}$ is a non-empty convex compact from $\mathbb{R}^N$.

System (4) allows standard form

$$\dot{x}(t) = Ax(t) + bu(t) + Cv(t), \; x(t_0) = x^0, \; C = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix},$$

with matrices A, b and M defined in (2). Note that the disturbance enters only half of the equations.

Under feedback control $\mathscr{U}(t,x) \in \mathscr{U}_{\mathrm{CL}}$ the system becomes a differential inclusion

$$\dot{x}(t) \in Ax(t) + b\mathscr{U}(t,x) + C\mathscr{Q}(t), \quad t \geqslant t_0, \tag{5}$$

and again its solutions exist on the interval $[t_0, t_1]$ due to properties of class $\mathscr{U}_{\mathrm{CL}}$.

Due to the disturbances, it may be impossible to guarantee exact damping of the chain. Instead we require that the control steers the system into a prescribed ε-neighborhood of the equilibrium.

[1] The problem of fully stopping a linear system with minimum intensity of control was considered in [1], [2] and by many other authors.

Problem 2: For a given $\varepsilon > 0$ find the solvability domain $\mathscr{W}_\varepsilon[t_0] \subseteq \mathbb{R}^n$ and a feedback control $\mathscr{U}_\varepsilon(t,x) \in \mathscr{U}_{\rm CL}$, such that all trajectories of (5) starting in $\mathscr{W}_\varepsilon[t_0]$ at time t_0 would satisfy $\|x(t_1)\| \leqslant \varepsilon$.

The set $\mathscr{W}_\varepsilon[t_0]$ is a weakly invariant set relative to the target. Such sets are crucial for constructing exact solution strategies or their approximations (for example, by using their ellipsoidal approximations as indicated below).

The last problem (2) may be also interpreted as a differential game [5]–[7].

III. The Solution

A. System without Uncertainty

In this Section we first consider the system in the absence uncertainty (Problem 1). Although this problem does not have any optimization criterion, in order to apply the dynamic programming techniques we require the control to minimize the deviation from the equilibrium state at the final time t_1. In particular, if this deviation is zero, then the chain is in the equilibrium and the control solves Problem 1. We introduce a corresponding *value function*

$$V(t,x) = \min_{\mathscr{U}\in\mathscr{U}_{\rm CL}} \max_{x(\cdot)\in\mathscr{X}_{\mathscr{U}}(\cdot)} \|x(t_1)\|^2. \tag{6}$$

Here $\mathscr{X}_{\mathscr{U}}(\cdot)$ is the trajectory tube of solutions to the differential inclusion (3) with fixed closed-loop control $\mathscr{U}(t,x)$ and initial condition $x(t) = x$.

The value function actually depends not only on the initial state (t,x), but also on the final time t_1 and the terminal functional, which is in our case $\varphi(x) = \|x\|^2 = d^2(x,\{0\})$. Extended notation which reflects this dependence is

$$V(t,x) = V(t,x;t_1,\varphi(\cdot)) = \min_{\mathscr{U}\in\mathscr{U}_{\rm CL}} \max_{x(\cdot)\in\mathscr{X}_{\mathscr{U}}(\cdot)} \varphi(x(t_1)).$$

Using the above notation we formulate the *principle of optimality* as a semigroup property for a generalized dynamic system

$$V(t,x;t_1,\varphi(\cdot)) = V(t,x;\tau,V(\tau,\cdot;t_1,\varphi(\cdot))) \tag{7}$$

for all $t \leqslant \tau \leqslant t_1$. From (7) it follows that the pair (t,x) contains all information about state of the system, and thus we can write down the fundamental equation of dynamic programming.

Theorem 1: The value function $V(t,x)$ is a classical solution to the Hamilton–Jacobi–Bellman (HJB) equation

$$V_t + \min_{u_{\min}\leqslant u\leqslant u_{\max}} \langle V_x, Ax + bu\rangle = 0, \quad t < t_1, \tag{8}$$

with initial condition $V(t_1,x) = \|x\|^2$.

The proof is based on using convex analysis [11] (see [3]).

The solution of the Problem 1 may be expressed through the value function if the latter is known. The solvability domain is then the zero level set of the value function:

$$\mathscr{W}[t] = \{\, x \mid V(t,x) \leqslant 0\}, \tag{9}$$

and the optimal control synthesis is the set of minimizers in (8):

$$\mathscr{U}^*(t,x) = \operatorname*{Arg\,min}_{u_{\min}\leqslant u\leqslant u_{\max}} \langle V_x, bu\rangle =$$
$$= \operatorname*{Arg\,min}_{u_{\min}\leqslant u\leqslant u_{\max}} V_{x_{2N}} u = \begin{cases} u_{\min}, & V_{x_{2N}} > 0; \\ u_{\max}, & V_{x_{2N}} < 0; \\ [u_{\min}, u_{\max}], & V_{x_{2N}} = 0. \end{cases}$$

Note that the control only takes extreme values $u_{\min}$, $u_{\max}$, except for states x inside the solvability domain $\mathscr{W}[t]$ and some degenerate points outside $\mathscr{W}[t]$ where $V_{x_{2N}} = 0$.

For simplifying numerical procedures note that the value function may be expressed through the solvability domain $\mathscr{W}[t]$. Applying convex analysis techniques we get (see [3]):

$$V(t,x) = d^2(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t]). \tag{10}$$

This may be further detailed as

$$\begin{aligned} &d^2(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t]) = \\ &\quad= \max_{\ell\in\mathbb{R}^n} \langle \ell, x\rangle - \rho(\ell \mid \mathscr{W}[t]) - \frac{1}{4}\left\|\mathrm{e}^{(t-t_1)A}\ell\right\|^2 = \\ &\quad= \langle \ell^0, x\rangle - \rho(\ell^0 \mid \mathscr{W}[t]) - \frac{1}{4}\left\|\mathrm{e}^{(t-t_1)A}\ell^0\right\|^2, \end{aligned} \tag{11}$$

where $\rho(\ell \mid \mathscr{W}[t])$ is the support function of $\mathscr{W}[t]$ in the direction ℓ (see [11]):

$$\rho(\ell \mid \mathscr{W}[t]) = \max_{x\in\mathscr{W}[t]} \langle \ell, x\rangle, \tag{12}$$

and ℓ^0 is the maximizer which is unique due to strong convexity of the function being maximized. For the present problem the support function (12) is

$$\rho(\ell \mid \mathscr{W}[t]) = \int_t^{t_1} [u_{\max}\cdot(s_{2N}(\tau))_+ + u_{\min}\cdot(s_{2N}(\tau))_-]\,\mathrm{d}\tau, \tag{13}$$

where $a_- = \min\{a,0\}$, $a_+ = \max\{a,0\}$, and $s(t)$ is the solution to the *adjoint equation*

$$\dot{s}(t) = -A^T s(t), \quad s(t_1) = \ell. \tag{14}$$

Then the optimal feedback control is

$$\mathscr{U}^*(t,x) = \begin{cases} u_{\min}, & \ell_n^0 > 0; \\ u_{\max}, & \ell_n^0 < 0; \\ [u_{\min}, u_{\max}], & \ell_n^0 = 0. \end{cases} \tag{15}$$

Theorem 2: The feedback control defined by (11)–(15), belongs to the class $\mathscr{U}_{\rm CL}$ of admissible closed-loop controls, and with the solvability domain defined by its support function (13) constitutes a solution to Problem 1.

The hardest computational part of the above solution is the optimization problem in (11). The latter may be solved much simpler if the exact solvability domain $\mathscr{W}[t]$ is replaced by an approximation $\mathscr{Z}[t]$ of a fairly simple structure.

We will further use *ellipsoids* to approximate the solvability domain:

$$\mathscr{E}(q,Q) = \left\{\, x \in \mathbb{R}^n \;\middle|\; \left\langle x - q, Q^{-1}(x - a)\right\rangle \leqslant 1\right\}.$$

Here q is the center of the ellipsoid, and Q is its configuration matrix (its eigenvectors and eigenvalues determine the orientation and size of ellipsoid). The support function of ellipsoid is $\rho(\,\ell \mid \mathscr{E}(q,Q)) = \langle \ell, q\rangle + \langle \ell, Q\ell\rangle^{\frac{1}{2}}$.

For the purposes of control synthesis, the following property of weakly invariant sets is important [3]:

Theorem 3: Let $\mathscr{Z}[t]$ be a weakly invariant set-valued mapping, such that $\rho(\,\ell \mid \mathscr{Z}[t])$ is differentiable in t for each $\ell \in \mathbb{R}^{2N}$. Then the function $Z(t,x) = d^2(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{Z}[t])$ satisfies differential inequality

$$\min_{u_{\min}\leqslant u\leqslant u_{\max}} \frac{\mathrm{d}Z(t,x(t))}{\mathrm{d}t} = \\ = Z_t + \min_{u_{\min}\leqslant u\leqslant u_{\max}} \langle Z_x, Ax+bu\rangle \leqslant 0. \quad (16)$$

As a consequence of (16), for the closed-loop control defined as a set of minimizers in (16)

$$\mathscr{U}_{\mathscr{Z}}(t,x) = \mathop{\mathrm{Arg\,min}}_{u_{\min}\leqslant u\leqslant u_{\max}} \langle Z_x, bu\rangle \quad (17)$$

the following property holds: if the initial point $x(t_0)$ of trajectory of differential inclusion (3) is inside $\mathscr{Z}[t_0]$, then the rest of trajectory also lies inside the tube $\mathscr{Z}[t_0]$. This is true because the distance from $x(t)$ to $\mathscr{Z}[t]$ is a non-increasing function.

Hence, if an internal approximation of the solvability domain is weakly invariant, then the closed-loop control (17) damps the chain from all initial states in $\mathscr{Z}[t_0]$. The control may be calculated using formulae (11)–(15) with $\mathscr{W}[t]$ replaced by $\mathscr{Z}[t]$. Such feedback strategy may be interpreted as “aiming” at the tube $\mathscr{Z}[t]$.

Consider the following differential equations for the parameters of internal ellipsoidal approximation $\mathscr{E}(x^*(t), X_-(t))$ [15]–[17]:

$$\dot{x}^*(t) = Ax^*(t) + bp, \quad x^*(t_1) = 0; \quad (18)$$
$$\dot{X}_-(t) = AX_-(t) + X_-(t)A^T + \\ X_-^{\frac{1}{2}}(t)S(t)(bPb^T)^{\frac{1}{2}} + (bPb^T)^{\frac{1}{2}}S^T(t)X_-^{\frac{1}{2}}(t), \quad X_-(t_1) = 0;$$
$$S(t)P^{\frac{1}{2}}B^Ts(t) = \lambda(t)X_-^{\frac{1}{2}}s(t), \quad S^T(t)S(t) = I.$$

Here $p = \frac{1}{2}(u_{\min} + u_{\max})$ and $P = \frac{1}{4}(u_{\max} - u_{\min})^2$ are the parameters of ellipsoid $\mathscr{E}(p,P) = \mathscr{P} = [u_{\min}, u_{\max}]$.

The ellipsoidal approximation defined in (18) is tangent to the solvability domain in the direction determined by the adjoint system (14) (which in its turn depends on direction $\ell \in \mathbb{R}^{2N}$):

$$\rho(\,s(t) \mid \mathscr{E}(x^*(t), X_-(t))) = \rho(\,s(t) \mid \mathscr{W}[t]). \quad (19)$$

It follows from (19) that the union of ellipsoidal approximations over all directions $\ell \in \mathbb{R}^{2N}$ is exactly the solvability domain:

$$\mathscr{W}[t] = \bigcup \{\,\mathscr{E}(x^*(t), X_-(t)) \mid \|\ell\| = 1\}.$$

For ellipsoidal approximation, the maximizing ℓ^0 in (11) may be calculated as

$$\ell^0 = 2\lambda(X_- + \lambda F)^{-1}(x - x^*), \quad (20)$$
$$F = \mathrm{e}^{(t-t_1)A^T}\mathrm{e}^{(t-t_1)A},$$

where λ is the unique non-negative root of

$$\big\langle (X_- + \lambda F)^{-1}(x-x^*), X_-(X_- + \lambda F)^{-1}(x-x^*)\big\rangle = 1, \quad (21)$$

or $\ell^0 = 0$ if (21) does not have positive roots.

It is possible to save computation time by avoiding solving the equation (21) if in the definition of $Z(t,x)$ one uses the metric defined by the matrix $X_-(t)$ (see [18]).

B. System with Uncertainty

The solution presented in the previous paragraph may be extended to the case of unknown disturbances (Problem 2). Below we provide this solution, highlighting the changed caused by the presence of uncertainty.

The value function is again defined by (6), with $\mathscr{X}_{\mathscr{U}}(\cdot)$ being the solution tube of the differential inclusion (5) with fixed feedback control $\mathscr{U}(t,x)$ and initial state $x(t) = x$.

The value function satisfies the principle of optimality (7), which allows to formulate the fundamental equation of dynamic programming. In general case, the value function here is not differentiable everywhere, but it satisfies the dynamic programming equation in the sense of directional derivatives which do exist since the value function is convex[2]

Theorem 4: The value function $V(t,x)$ is a solution to the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation

$$V_t + \min_{u_{\min}\leqslant u\leqslant u_{\max}} \max_{v\in\mathscr{Q}} \langle V_x, Ax+bu+Cv\rangle = 0 \quad (22)$$

$t < t_1$, with initial condition

$$V(t_1,x) = (\max\{\|x\| - \varepsilon, 0\})^2.$$

Note that the HJB equation (8) is a particular case of (22) with $\mathscr{Q} = \{0\}$.

The solvability domain $\mathscr{W}[t]$ is a zero level set of the value function (9). Unfortunately, unlike the problem without uncertainty it is not possible to express the value function via the solvability domain, as in (10). However, for the purposes of control synthesis the following estimate is sufficient [3]:

$$V(t,x) \leqslant W(t,x) = d^2\Big(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t]\Big).$$

If we substitute this estimate into the HJBI equation instead of the value function, we get a differential inequality (cf. (16))

$$\min_{u_{\min}\leqslant u\leqslant u_{\max}} \max_{v\in\mathscr{Q}} \frac{\mathrm{d}W(t,x)}{\mathrm{d}t} = W_t + \\ + \min_{u_{\min}\leqslant u\leqslant u_{\max}} \max_{v\in\mathscr{Q}} \langle W_x, Ax+bu+Cv\rangle \leqslant 0. \quad (23)$$

We define the feedback strategy $\mathscr{U}_{\mathscr{W}}(t,x)$ as the set of minimizers in (23):

$$\mathscr{U}_{\mathscr{W}}(t,x) = \mathop{\mathrm{Arg\,min}}_{u_{\min}\leqslant u\leqslant u_{\max}} \\ \Big\langle \frac{\partial}{\partial x} d^2\Big(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t]\Big), Bu\Big\rangle. \quad (24)$$

[2]One may also interpret the value function as a generalized solution to the HJBI equation — viscosity solutions introduces by M. G. Crandall and P.-L. Lions [20], [21], min-max solutions by A. I. Subbotin [22].

If the solvability domain is known, the control synthesis (24) may be calculated using (15), (11).

Equations (23), (24) will be true if one replaces the solvability domain $\mathscr{W}[t]$ with any weakly invariant set-valued mapping. Among them there is an internal ellipsoidal approximation $\mathscr{E}(x^*(t), X_-(t))$ with parameters determined by the following ODEs [15]:

$$\dot{x}^*(t) = Ax^*(t) + bp + Cq, \quad x^*(t_1) = 0;$$
$$\dot{X}_-(t) = AX_-(t) + X_-(t)A^T -$$
$$-\pi(t)X_-(t) - \pi^{-1}(t)CQC^T +$$
$$X_-^{\frac{1}{2}}(t)S(t)(bPb^T)^{\frac{1}{2}} + (bPb^T)^{\frac{1}{2}}S^T(t)X_-^{\frac{1}{2}}(t),$$
$$X_-(t_1) = \varepsilon^2 I;$$
$$S(t)P^{\frac{1}{2}}B^T s(t) = \lambda(t)X_-^{\frac{1}{2}}s(t), \quad S^T(t)S(t) = I;$$
$$\pi(t) = \frac{\langle s(t), CQC^T s(t)\rangle^{\frac{1}{2}}}{\langle s(t), X_-(t)s(t)\rangle^{\frac{1}{2}}}.$$

We assume the set of possible values of disturbances $\mathscr{Q}$ is also an ellipsoid $\mathscr{Q} = \mathscr{E}(q, Q)$. If this is not the case, it is necessary to use ellipsoid circumscribed around $\mathscr{Q}$.

For each direction $\ell \in \mathbb{R}^{2N}$ there exists time τ such that the ellipsoidal approximation is tangent to the solvability domain in direction ℓ (i.e. (14) holds) on the interval $[t_1-\tau, t_1]$. The closure of the union of all ellipsoidal approximations is equal to the solvability domain:

$$\mathscr{W}[t] = \mathrm{cl}\bigcup\{\mathscr{E}(x^*(t), X_-(t)) \mid \|\ell\| = 1\}.$$

After calculating the ellipsoidal approximation the control synthesis is calculated using (15), (20), (21).

C. A Specific Disturbance

An important particular case of the problem with uncertainty is when the disturbance only enters the last equation of (4), i.e. $v_1 = \ldots = v_{N-1} = 0$, $v_N \in [v_{\min}, v_{\max}]$.

In this case it is easy to see that the HJBI equation (22)

$$V_t + \min_{u_{\min}\leqslant u\leqslant u_{\max}} \max_{v_N\in[v_{\min},v_{\max}]} \langle V_x, Ax + b(u+v_N)\rangle = 0$$

is equivalent (i.e. has the same solution) to the following HJB equation:

$$V_t + \min_{u_{\min}+v_{\max}\leqslant u\leqslant u_{\max}+v_{\min}} \langle V_x, Ax + bu\rangle = 0.$$

Therefore, this particular case reduces to a problem without uncertainty with $\mathscr{P} = [u_{\min} + v_{\max}, u_{\max} + v_{\min}]$.

IV. Examples

The algorithm described above has been implemented in Matlab. We have used the ellipsoidal toolbox [24] to calculate the internal ellipsoidal approximations of the solvability domain and introduce fairly simple algorithms for on-line calculation of the desired control strategies. The text is accompanied by computer animation and numerical simulations for systems of realistically important dimension subjected to various types of disturbances.

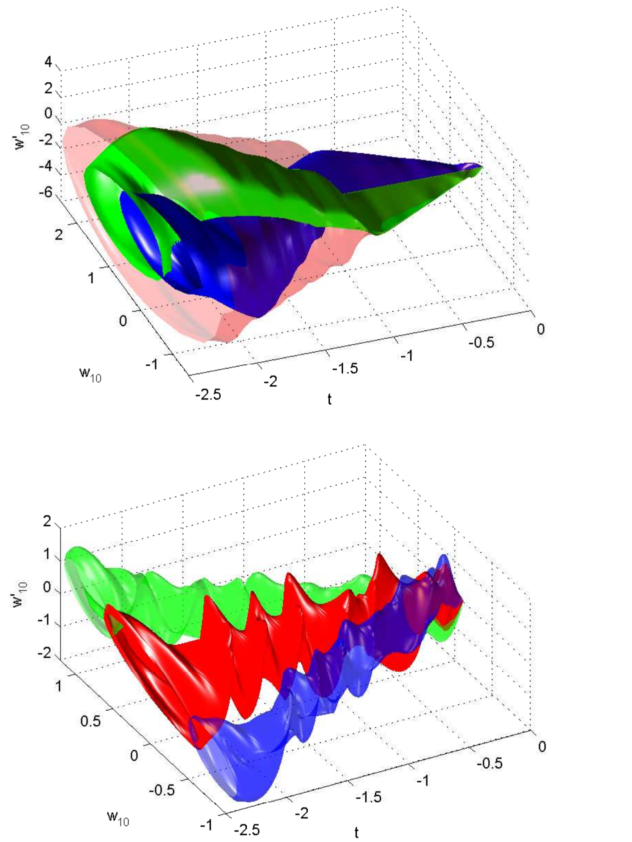

Fig. 2. Projections of solvability tubes for Example 1. Above: exact tubes, below: ellipsoidal approximations. Red: $\mathscr{P} = [-1, 1]$, green: $\mathscr{P} = [0, 1]$, blue: $\mathscr{P} = [-1, 0]$.

Example 1: The chain has $N = 10$ springs with stiffness coefficients $k_i \equiv N$ and loads $m_i \equiv \frac{1}{N}$. We consider three different bounds on control: $\mathscr{P}_1 = [-1, 1]$ (control force may be applied both up and down), $\mathscr{P}_2 = [0, 1]$ (only down) and $\mathscr{P}_3 = [-1, 0]$ (only up). There are no disturbances.

Fig. 2, upper plot, shows projections of the corresponding solvability tubes $\mathscr{W}_1[\cdot]$ (red), $\mathscr{W}_2[\cdot]$ (green) and $\mathscr{W}_3[\cdot]$ (blue). Note that, as expected, $\mathscr{W}_1$ is larger than the union of $\mathscr{W}_2$ and $\mathscr{W}_3$. The tubes were calculated using (13), (14).

Fig. 2, lower plot, shows projections of ellipsoidal approximations (for a specific direction $\ell \in \mathbb{R}^{20}$) of the tubes above, $\mathscr{E}_i \subseteq \mathscr{W}_i$. Although $\mathscr{W}_2, \mathscr{W}_3 \subseteq \mathscr{W}_1$, this inclusion is not true for the ellipsoidal approximations. In fact, there are times when $\mathscr{E}_1$ does not even intersect with $\mathscr{E}_2$, $\mathscr{E}_3$.

Example 2: Consider a chain with $N = 4$ springs with $k_i \equiv 1$, $m_i \equiv 1$. The lowest eigenfrequency of this chain is $\omega_0 \approx 0{,}35$. We introduce a disturbance $v(t) = 0{,}5\sin\omega_0 t$, which is intended to "shake" the chain at the resonant frequency. The control with hard bound $\mathscr{P} = [0, 1]$ (acting only down) is to mitigate the disturbance and damp the chain.

Fig. 3 shows the disturbance $v(t)$ (first plot) and the history of $\|x(t)\|$ (second plot) when the control is turned off. The amplitude of oscillations clearly grows with time. The corresponding control (third plot) indeed effectively damps the oscillations (see the trajectory of $\|x(t)\|$ in the fourth plot).

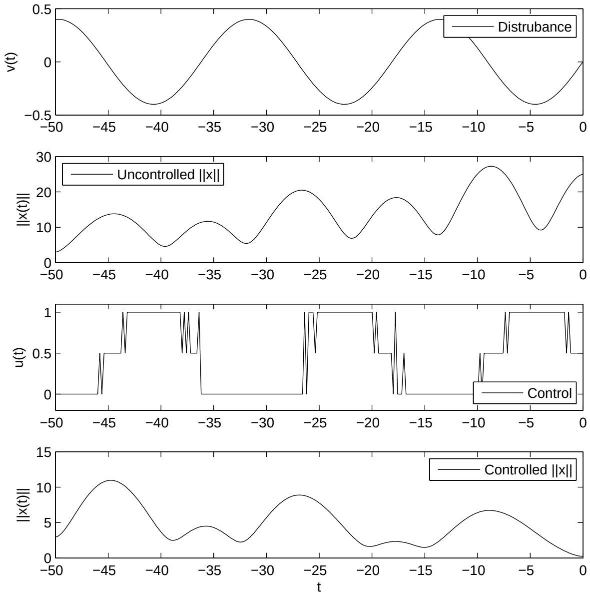

Fig. 3. Illustration for Example 2

V. Conclusions

This paper indicates worked out techniques for solving problems of closed-loop control for oscillating systems of high dimension subjected to unknown but bounded disturbances. While using the approach of reachability and weakly invariant sets (which also coincide with "Krasovski bridges"), this paper indicates effective schemes for calculating closed-loop solution strategies. These are based on using internal ellipsoidal approximations of weakly invariant solvability sets and produce effective algorithmic schemes. The computerized solution versions allow to test the effectiveness of available closed-loop controls over various classes of loads, disturbances and broad ranges of initial conditions.

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