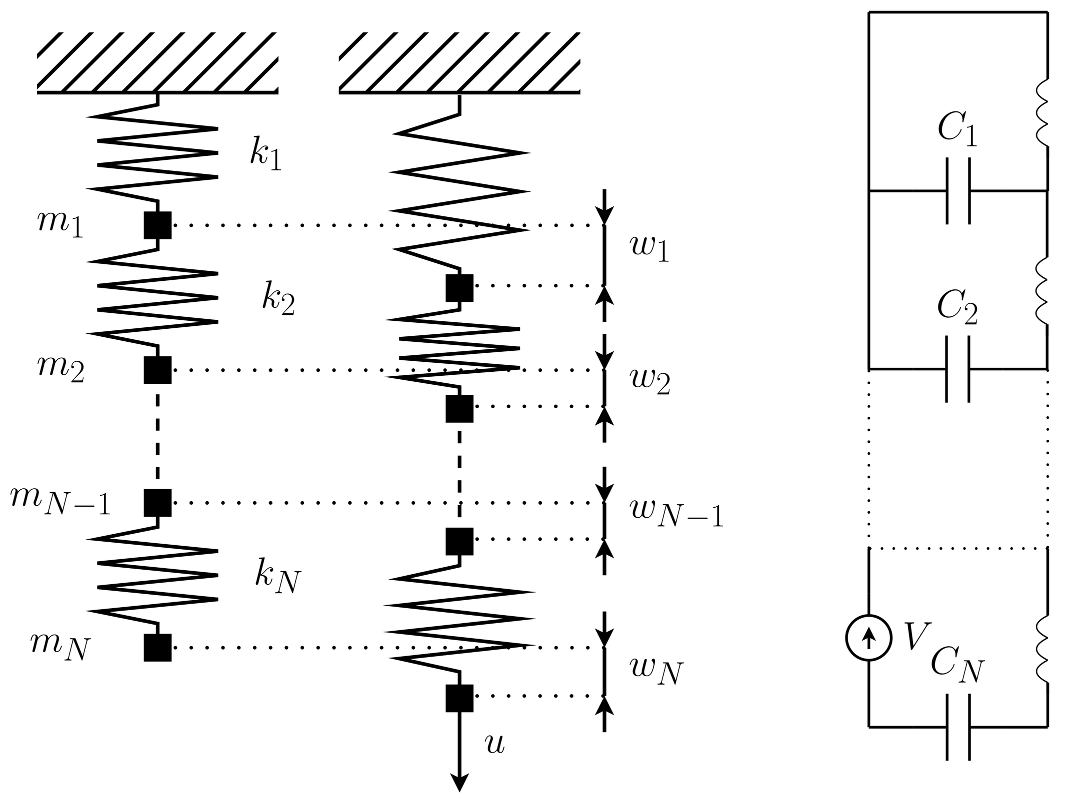
Reachability Approaches and Ellipsoidal Techniques for Closed-Loop Control of Oscillating Systems under Uncertainty

A. N. Daryin, A. B. Kurzhanski and I. V. Vostrikov · Moscow State (Lomonosov) University

Abstract

Within the range of important applied control problems under present investigation is the one of feedback control of oscillating systems of high dimensions. The given paper indicates effective schemes for solving such problems for systems subjected to unknown but bounded disturbances including those that may also be in resonance with system. The selected approach is based on constructing weakly invariant sets (similar to "Krasovski bridges") which are then further used to design the specific solution strategies and also to investigate the damping possibilities of the controls under various types of loads and disturbances. The text is accompanied by computer animation and numerical simulations based on ellipsoidal tools for calculating reach sets. This allows to tackle problems of realistically high dimensions and solve them in limited time.

Model



$$\begin{cases} m_1\ddot{w}_1 = k_2\cdot(w_2-w_1) - k_1 w_1 + v_1, \\ m_i\ddot{w}_i = k_{i+1}\cdot(w_{i+1}-w_i) - k_i\cdot(w_i - w_{i-1}) + v_i, \quad i=\overline{2,N-1}, \\ m_N\ddot{w}_N = -k_N\cdot(w_N - w_{N-1}) + u + v_N. \end{cases}$$

u is control
v_i are disturbances

Normalized matrix form (with $x=(w,\dot{w})$):

$$\dot{x}(t) = Ax(t) + bu + Cv. \qquad (1)$$

This system is **completely controllable**.

If N is large, the dimension of the system is high.
(In our numerical experiments, N is up to 25, so that the dimension is 50).

Constraints

	Control	Disturbance
	$\mu_{\min} \leqslant u \leqslant \mu_{\max}$	$v \in \mathscr{Q}$ — convex compact
e.g.	• bilateral: $u\in[-\mu,\mu]$ • unilateral: $u\in[0,\mu]$, $u\in[-\mu,0]$	• in all equations • in the last equation only

Disturbance may be in the resonance with the system!

Control Problem

Class of closed-loop controls $\mathscr{U}_{\mathrm{CL}}$: set-valued mappings $\mathscr{U}(t,x)$

- convex compact values
- measurable in t
- upper semicontinuous in x.

(1) becomes a **differential inclusion**

$$\dot{x} \in Ax + b\mathscr{U}(t,x) + C\mathscr{Q} \qquad (2)$$

Problem: For a given $\varepsilon > 0$, find

- *backward reach set (solvability domain)* $\mathscr{W}[t_0] \subseteq \mathbb{R}^{2N}$
- closed-loop control $\mathscr{U}(t,x)$

s.t. all trajectories starting in $\mathscr{W}[t_0]$ satisfy $\|x(t_1)\| \leqslant \varepsilon$.

We look for solution IN GIVEN FINITE TIME $[t_0,t_1]$.

This is **not** a problem of asymptotic stabilization!

Dynamic Programming

The **value function**:

$$V(t,x) = V(t,x;t_1,V(t_1,\cdot)) = \min_{\mathscr{U}\in\mathscr{U}_{\mathrm{CL}}} \max_{x(\cdot)} \{\max\{0,\|x(t_1)\|-\varepsilon\} \mid x(t)=x\}$$

Principle of optimality: $V(t,x;t_1,V(t_1,\cdot)) = V(t,x;\tau,V(\tau,\cdot;t_1,V(t_1,\cdot)))$, $t\leqslant\tau\leqslant t_1$.

Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation:

$$V_t + \min_{u\in[\mu_{\min},\mu_{\max}]} \max_{v\in\mathscr{Q}} \langle V_x, Ax+bu+Cv\rangle = 0, \quad t<t_1, \quad V(t_1,x) = \max\{0,\|x\|-\varepsilon\}.$$

Solution/estimate ($d(x,\mathscr{X})$ is the Euclidian distance from point x to set X):

$$V(t,x) = d\left(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathscr{W}[t]\right) \qquad V(t,x) \leqslant d\left(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathscr{W}[t]\right)$$

without disturbance — under disturbance

Optimal closed-loop control strategy:

$$\mathscr{U}^*(t,x) = \operatorname*{Arg\,min}_{u\in[\mu_{\min},\mu_{\max}]} \langle V_x, bu\rangle = \begin{cases} \mu_{\min}, & V_{x_{2N}} > 0; \\ \mu_{\max}, & V_{x_{2N}} < 0; \\ [\mu_{\min},\mu_{\max}], & V_{x_{2N}} = 0. \end{cases}$$

Approximation

Finding and storing the exact solvability domain $\mathscr{W}[t]$ and the exact value function $V(t,x)$ for large N is computationally hard and consumes much memory.

A possible solution is:

- replace the solvability domain $\mathscr{W}[t]$ with an inner estimate $\mathscr{W}_-[t]$ of a simple structure;
- replace the value function $V(t,x)$ with an upper estimate $d\left(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathscr{W}_-[t]\right)$.

The latter leads to the strategy of "aiming" to $\mathscr{W}_-[t]$:

$$\mathscr{U}_{\mathscr{W}_-}(t,x) = \begin{cases} -\mu, & \ell^0_{2N} > 0 \\ \mu, & \ell^0_{2N} < 0 \\ [-\mu,\mu], & \ell^0_{2N} = 0 \end{cases}$$

where ℓ^0 is from $d(e^{(t_1-t)A}x, e^{(t_1-t)A}\mathscr{W}_-[t]) = \max_{\|\ell\|\leqslant 1}\{\langle \ell, x\rangle - \rho(\ell \mid \mathscr{W}_-[t])\} = \langle \ell^0, x\rangle - \rho\left(\ell^0 \mid \mathscr{W}_-[t]\right)$.

Ellipsoidal Approximation

Ellipsoids

$$\mathscr{E}(m,M) = \left\{ x \;\middle|\; \left\langle x-m, M^{-1}(x-m)\right\rangle \leqslant 1 \right\} \qquad \rho(\ell \mid \mathscr{E}(m,M)) = \langle m,\ell\rangle + \sqrt{\langle \ell, M\ell\rangle}$$

(A. B. Kurzhanski and I. Vályi, Ellipsoidal Calculus for Estimation and Control. Boston: Birkhäuser, 1997.)

Ellipsoidal approximation: $\mathscr{X}[t] = \mathscr{E}(x^*(t), X_-(t))$, where

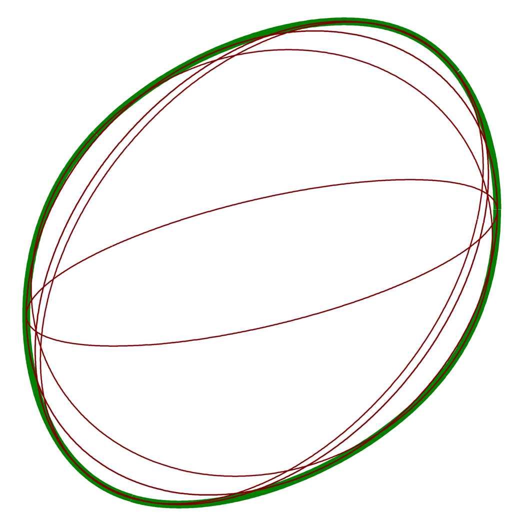
$$\dot{x}^*(t) = Ax^*(t) + bp + Cq, \quad x^*(t_1) = 0$$

$$\dot{X}_-(t) = AX_-(t) + X_-(t)A^T - \pi(t)X_-(t) - \pi^{-1}(t)CQC^T + X_-^{\frac{1}{2}}(t)S(t)(bPb^T)^{\frac{1}{2}} + (bPb^T)^{\frac{1}{2}}S^T(t)X_-^{\frac{1}{2}}(t)$$

$$X_-(t) = \varepsilon^2 I \qquad S(t)P^{\frac{1}{2}}B^T s(t) = \lambda(t)X_-^{\frac{1}{2}}s(t) \quad S^T(t)S(t) = I \qquad \pi(t) = \frac{\left\langle s(t), CQC^Ts(t)\right\rangle^{\frac{1}{2}}}{\langle s(t), X_-(t)s(t)\rangle^{\frac{1}{2}}}$$

$$p = \frac{1}{2}(\mu_{\min}+\mu_{\max}), \quad P = \frac{1}{4}(\mu_{\max}-\mu_{\min})^2, \quad \mathscr{Q} = \mathscr{E}(q,Q)$$

(A. B. Kurzhanski and P. Varaiya, "Ellipsoidal techniques for reachability analysis. Part II: Internal approximations. Box-valued constraints," Optimization methods and software, *vol. 17, pp. 177–237, 2002.)*



These approximations are **tight**:

$$\rho(s(t) \mid \mathscr{X}[t]) = \rho(s(t) \mid \mathscr{W}[t])$$

The control is calculated using

$$\ell^0 = X_-^{-1}[t]\cdot(x - x^*(t)).$$

Advantages

Ellipsoidal approximation is **efficient**:

- In n dimensional state space, an ellipsoid is described by only $O(n^2)$ values.
- Ellipsoidal approximation is calculated by solving ordinary differential equations.
- The control is calculated by solving a linear system.
- The algorithm allows parallel computations.
- Time-efficiency trade-off: the more ellipsoids you compute, the more accurate is the approximation.

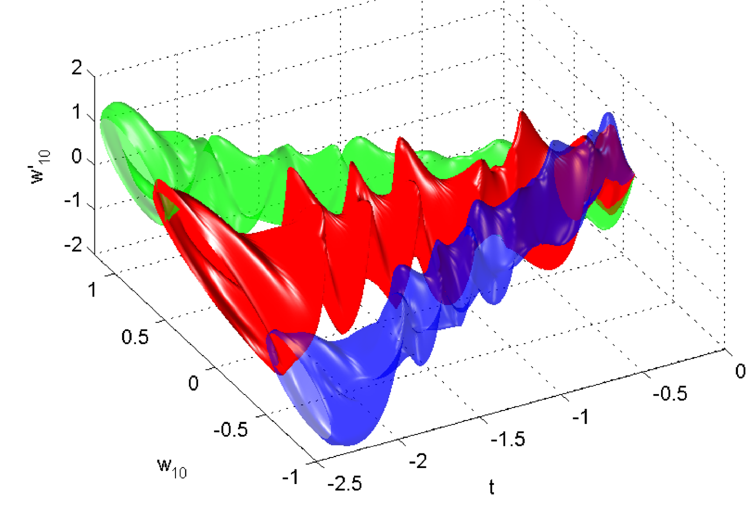
Using ellipsoidal approximations, one can **analyze relationships** between

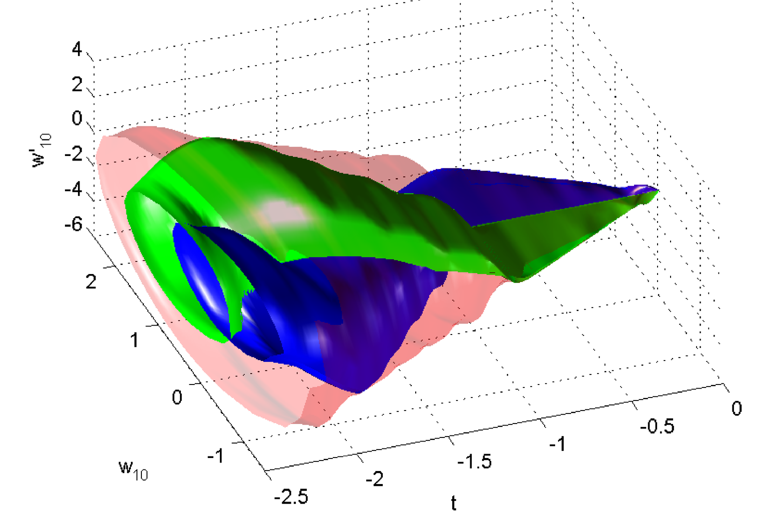
- bounds on control
- bounds on disturbance
- desired accuracy (ε)
- length of time interval.

The presented approach is a **general scheme** for control problems of high dimensions.

Numerical Experiments

The control algorithm is implemented using the **Ellipsoidal Toolbox**:
http://www.eecs.berkeley.edu/~akurzhan/ellipsoids/





See **computer simulations** on the laptop.