A Dynamic Programming Approach to the Impulse Control Synthesis Problem

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The Bang-Bang Control

$$\dot{x} = Ax + bu$$

$$|u| \leqslant \mu$$

Time-optimal control:

$$x(0) = x_0 \qquad x(\theta) = 0 \qquad \theta \to \min$$

$$u_0(t) = \mu \cdot \operatorname{sign} h_0(t)$$

The Impulse Control

$$\mathrm{d}x = Ax\,\mathrm{d}t + b\,\mathrm{d}U(t)$$

$$\frac{\mathrm{d}U}{\mathrm{d}t} = u$$ — generalized derivative

$$\operatorname{Var} U \leqslant \mu$$

Time-optimal control:

$$x(0) = x_0 \qquad x(\theta) = 0 \qquad \theta \to \min$$

$$u_0(t) = \sum_{i=1}^{k} \alpha_i \delta(t - t_i) \qquad \sum_{i=1}^{k} |\alpha_i| \leqslant \mu$$

$$t_0 \leqslant t_1 < t_2 < \ldots < t_k \leqslant \theta$$

The Problem

Problem 1. Minimize

$$J(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1+0)) \longrightarrow \inf,$$

over $U(\cdot) \in BV([t_0,t_1];\mathbb{R}^m)$ subject to

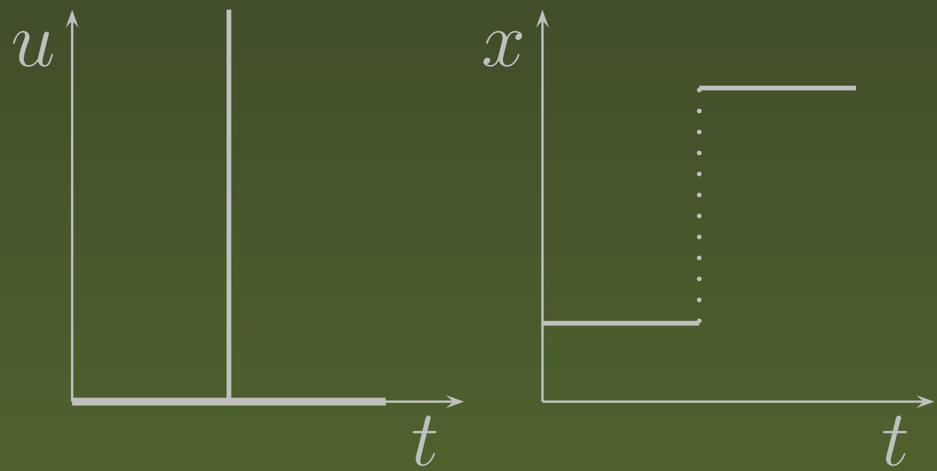
$$\mathrm{d}x(t) = A(t)x(t)\,\mathrm{d}t + B(t)\,\mathrm{d}U(t), \qquad t \in [t_0,t_1],$$

$$x(t_0-0) = x_0.$$

Important particular case:

$\varphi(x) = \delta(x \mid \{x_1\})$: steer from (t_0,x_0) to (t_1,x_1).

The Ideal Scheme



The Value Function

The minimum of $J(U(\cdot))$ with *fixed* initial position $x(t_0 - 0) = x_0$ is called the *value function*:

$$V(t_0, x_0) = V(t_0, x_0; t_1, \varphi(\cdot)).$$

Calculating the Value Function

Decompose the problem into two parts:

- find optimal $x_1 = x(t_1 + 0)$
- find optimal $U(\cdot)$ to steer from (t_0, x_0) to (t_1, x_1)

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - X(t_1, t_0)x_0 \rangle}{\|B'(\cdot)X'(t_1, \cdot)p\|_{C[t_0,t_1]}} \right\}.$$

$X(t, \tau)$ is the solution to

$$\partial X(t, \tau)/\partial t = A(t)X, \quad X(\tau, \tau) = I.$$

Calculating the Value Function II

$$X(t_1, t_0)x_0 + \int_{t_0}^{t_1} X(t_1, t)B(t)\,\mathrm{d}U(t) = x_1$$

$$\langle p, x_1 - X(t_1, t_0)x_0 \rangle = \int_{t_0}^{t_1} \langle B'(t)X'(t_1, t)p,\, \mathrm{d}U(t) \rangle \leqslant$$

$$\leqslant \left[\max_{t \in [t_0, t_1]} B'(t)X'(t_1, t)p \right] \cdot \operatorname*{Var}_{[t_0, t_1]} U(\cdot)$$

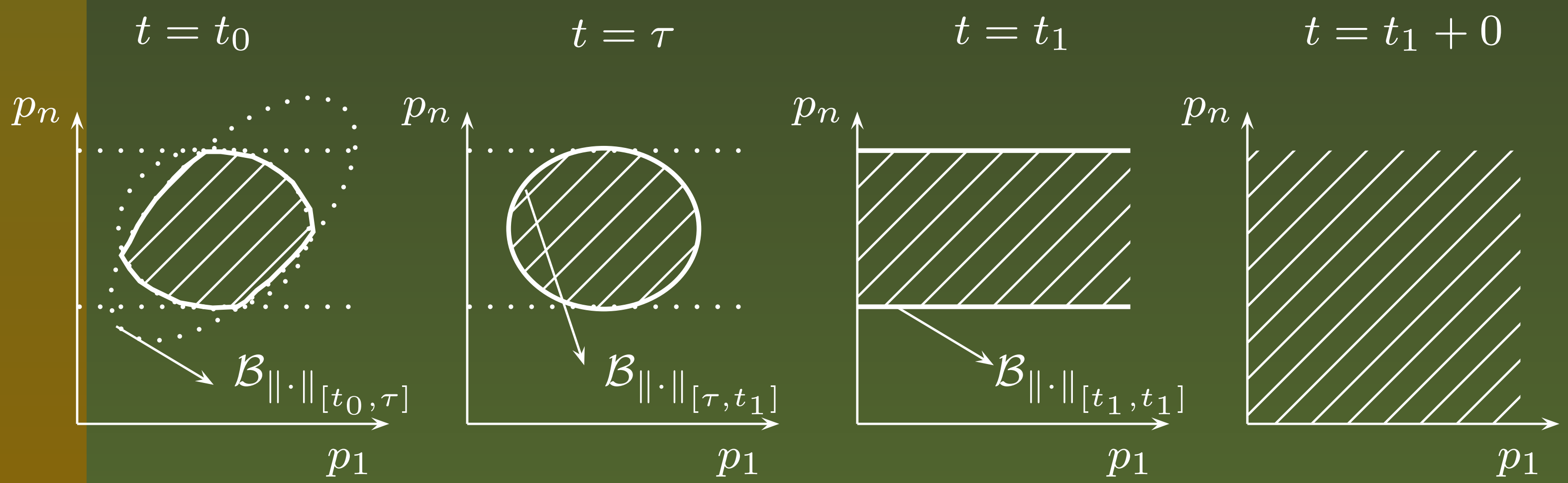
$$\inf \operatorname*{Var}_{[t_0, t_1]} U(\cdot) = \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - X(t_1, t_0)x_0 \rangle}{\|B'(\cdot)X'(t_1, \cdot)p\|_{C[t_0, t_1]}}$$

The Conjugate of the Value Function

The value function is convex and its conjugate equals

$$V^*(t_0, p) = \varphi^*(X'(t_0, t_1)p) + \delta\left(X'(t_0, t_1)p \;\middle|\; \mathcal{B}_{\|\cdot\|_{[t_0,t_1]}} \right)$$

where $\|p\|_{[t_0,t_1]} = \|B'(t)X'(t_1, \cdot)p\|_{C[t_0,t_1]}$.



Calculating the Conjugate

$$V(t_0,x_0) = \inf_{x_1\in\mathbb{R}^n}\left\{\varphi(x_1) + \sup_{p\in\mathbb{R}^n}\frac{\langle p, x_1 - X(t_1,t_0)x_0\rangle}{\|B'(\cdot)X'(t_1,\cdot)p\|_{C[t_0,t_1]}}\right\}.$$

$$V(t_0,x_0) = \inf_{x_1\in\mathbb{R}^n}\left\{\varphi(x_1) + \sup_{p\in\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}}\langle p, x_1 - X(t_1,t_0)x_0\rangle\right\}.$$

$$V(t_0,x_0) = \sup_{p\in\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}}\inf_{x_1\in\mathbb{R}^n}\{\varphi(x_1) + \langle p, x_1\rangle - \langle p, X(t_1,t_0)x_0\rangle\}.$$

$$V(t_0,x_0) = \sup_{p\in\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}}\{-\varphi^*(-p) - \langle p, X(t_1,t_0)x_0\rangle\}.$$

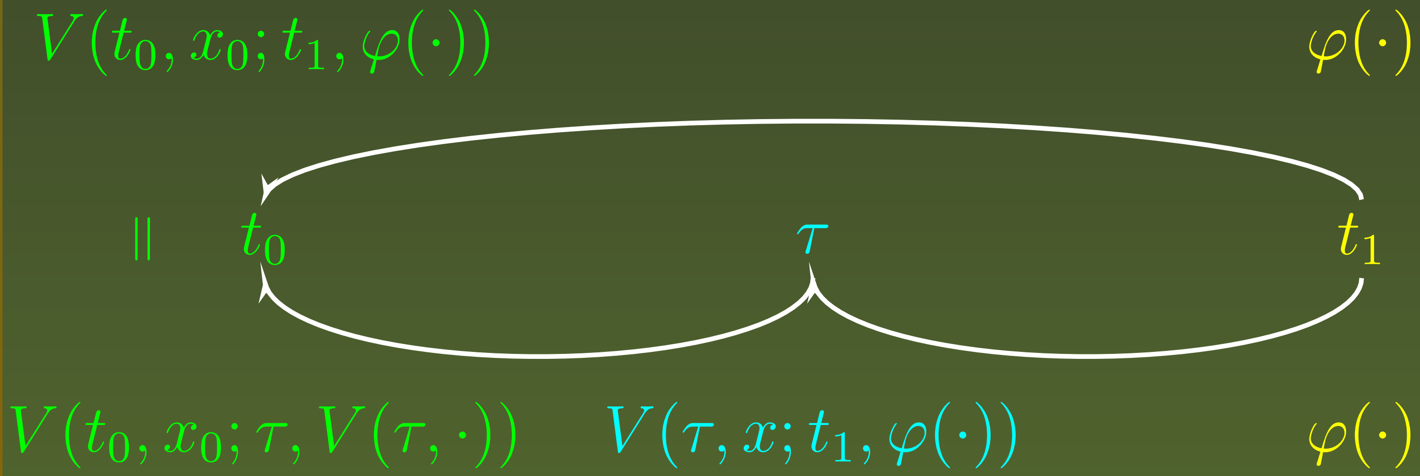
$$V(t_0,x_0) = \sup_{p\in\mathbb{R}^n}\left\{\varphi^*(p) - \langle p, X(t_1,t_0)x_0\rangle - \delta\left(p \;\middle|\; \mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}\right)\right\}.$$

The Principle of Optimality

The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the *Principle of Optimality*

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))),$$

where $\tau \in [t_0, t_1]$.



The Principle of Optimality II

In the general case $V(t_1, x; t_1, \varphi(\cdot)) \leqslant \varphi(x)$, since

$$V^*(t_1, p) = \varphi^*(p) + \delta(\, B(t_1)p \mid \mathcal{B}_1).$$

For example, if $\varphi(x) = \delta(\, x \mid \{x_1\})$ and $B(t_1) = I$, then

$$V(t_1, x; t_1, \varphi(\cdot)) = \|x - x_1\|.$$

However, due to the Principle of Optimality,

$$V(t_1, x; t_1, \varphi(\cdot)) = V(t_1, x; t_1, V(t_1, \cdot; t_1, \varphi(\cdot))).$$

Dynamic Programming Equation

The value function it is the viscosity solution to the Hamilton–Jacobi–Bellman equation:

$$\min\{H_1(t,x,V_t,V_x),\ H_2(t,x,V_t,V_x)\} = 0,$$

$$V(t_1,x) = V(t_1,x;t_1,\varphi(\cdot)).$$

$$H_1(t,x,\xi_t,\xi_x) = \xi_t + \langle \xi_x, A(t)x\rangle,$$

$$H_2(t,x,\xi_t,\xi_x) = \min_{u\in S_1}\langle \xi_x, B(t)u\rangle + 1 = -\left\|B^T(t)\xi_x\right\| + 1.$$

The Control Structure

$$H_1(t,x) = 0 \longleftarrow (t,x) \longrightarrow H_2(t,x) = 0$$

$\downarrow$ (from $H_1(t,x) = 0$)

wait

$$dU(t) = 0$$

$\downarrow$ (from $H_2(t,x) = 0$)

choose jump direction

$$d = -B'(t)V_x$$

$\downarrow$

choose jump amplitude

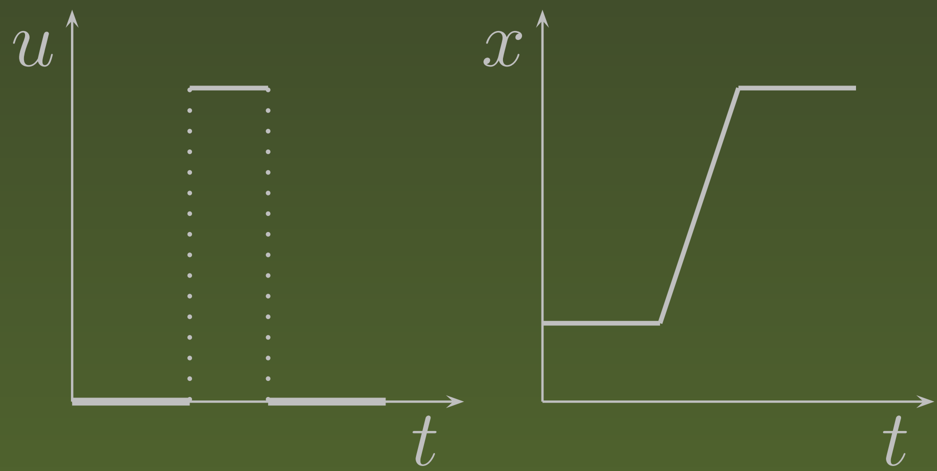
$$\min \alpha \geqslant 0 : H_1(t, x + \alpha d) = 0$$

$\downarrow$

jump

$$U(\tau) = \alpha \cdot d \cdot \chi(\tau - t)$$

The Realistic Scheme



The Additional Constraint Approach

Introduce a hard bound on the control, $u(t) \in \mathcal{B}_\mu$, and consider the corresponding problem:

Problem 2. Minimize

$$J_\mu(u(\cdot)) = \int_{t_0}^{t_1} \|u(t)\| \, \mathrm{d}t + \varphi(x(t_1)) \to \inf,$$

over $u(\cdot) \in L_1([t_0, t_1]; \mathbb{R}^m)$ subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,$$

$$\|u(t)\| \leqslant \mu.$$

The Value Function

The minimum of $J_\mu(U(\cdot))$ with *fixed* initial position $x(t_0) = x_0$ is called the *value function*:

$$V_\mu(t_0, x_0) = V_\mu(t_0, x_0; t_1, \varphi(\cdot)).$$

The value function of the Problem 2 is the viscosity solution to the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V_\mu}{\partial t} + \min_{u \in \mu \mathcal{B}_1} \left\{ \left\langle \frac{\partial V_\mu}{\partial x}, A(t)x(t) + B(t)u \right\rangle + \|u\| \right\} = 0,$$

$$V_\mu(t_1, x) = \varphi(x).$$

Calculating the Value Function

The value function of the Problem 2 is

$$V_\mu(t,x) = \sup_{p\in\mathbb{R}^n}\Big\{\langle p, X(t_1,t_0)x_0\rangle - \\ - \mu\int_t^{t_1}(\|B'(\tau)X'(t_1,\tau)p\| - 1)_+\,\mathrm{d}\tau - \varphi^*(p)\Big\},$$

and its conjugate in x is

$$V_\mu^*(t,p) = \varphi^*(p) + \mu\int_t^{t_1}(\|B'(\tau)X'(t_1,\tau)p\| - 1)_+\,\mathrm{d}\tau.$$

The Convergence of V_μ

As $\mu \to \infty$,

$$V_\mu(t,x) \to V(t,x), \quad V_\mu^*(t,x) \to V^*(t,x).$$

Under certain assumptions

$$0 \leqslant V_\mu(t,x) - V(t,x) = O(\mu^{-1}).$$

The HJB equation for Problem 1 may also be derived in the limit from the one for Problem 2, as $\mu \to \infty$.

The Control Synthesis

The optimal feedback control strategy for Problem 2 is the minimizer in the HJB equation:

$$\mathcal{U}_\mu^*(t,x) = \begin{cases} \{0\}, & \|\zeta\| < 1; \\ [0, -\mu\zeta], & \|\zeta\| = 1; \\ \left\{ -\mu \dfrac{\zeta}{\|\zeta\|} \right\}, & \|\zeta\| > 1, \end{cases} \qquad \zeta = B'(t)\frac{\partial V_\mu}{\partial x}$$

(in points where $V_\mu(t,x)$ is differentiable in x).

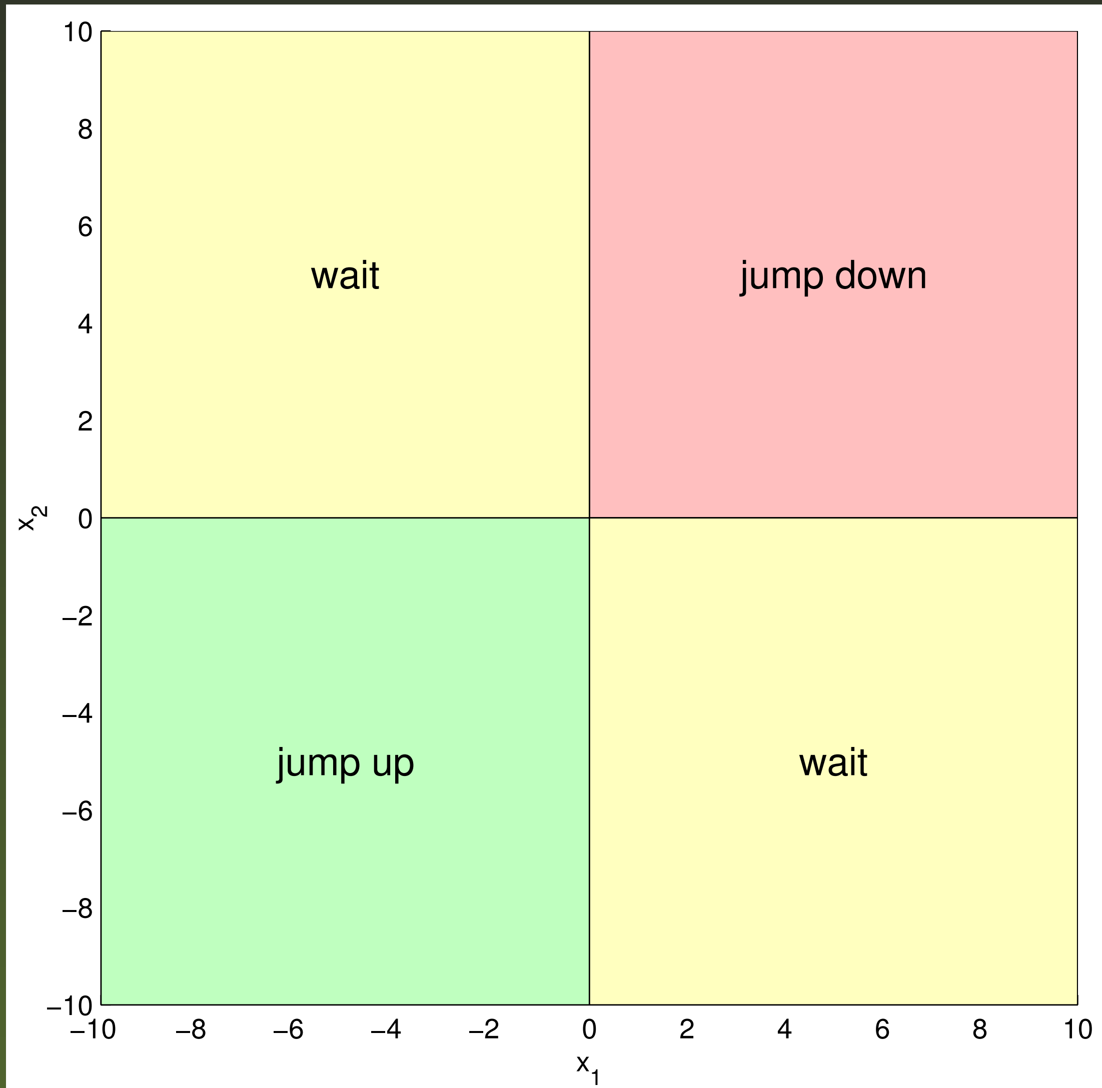
Example: The Problem

$$\begin{cases} \mathrm{d}x_1(t) = x_2(t)\,\mathrm{d}t, \\ \mathrm{d}x_2(t) = -x_1(t)\,\mathrm{d}t + \mathrm{d}U(t), \end{cases} \qquad 0 \leqslant t \leqslant \tfrac{\pi}{2},$$

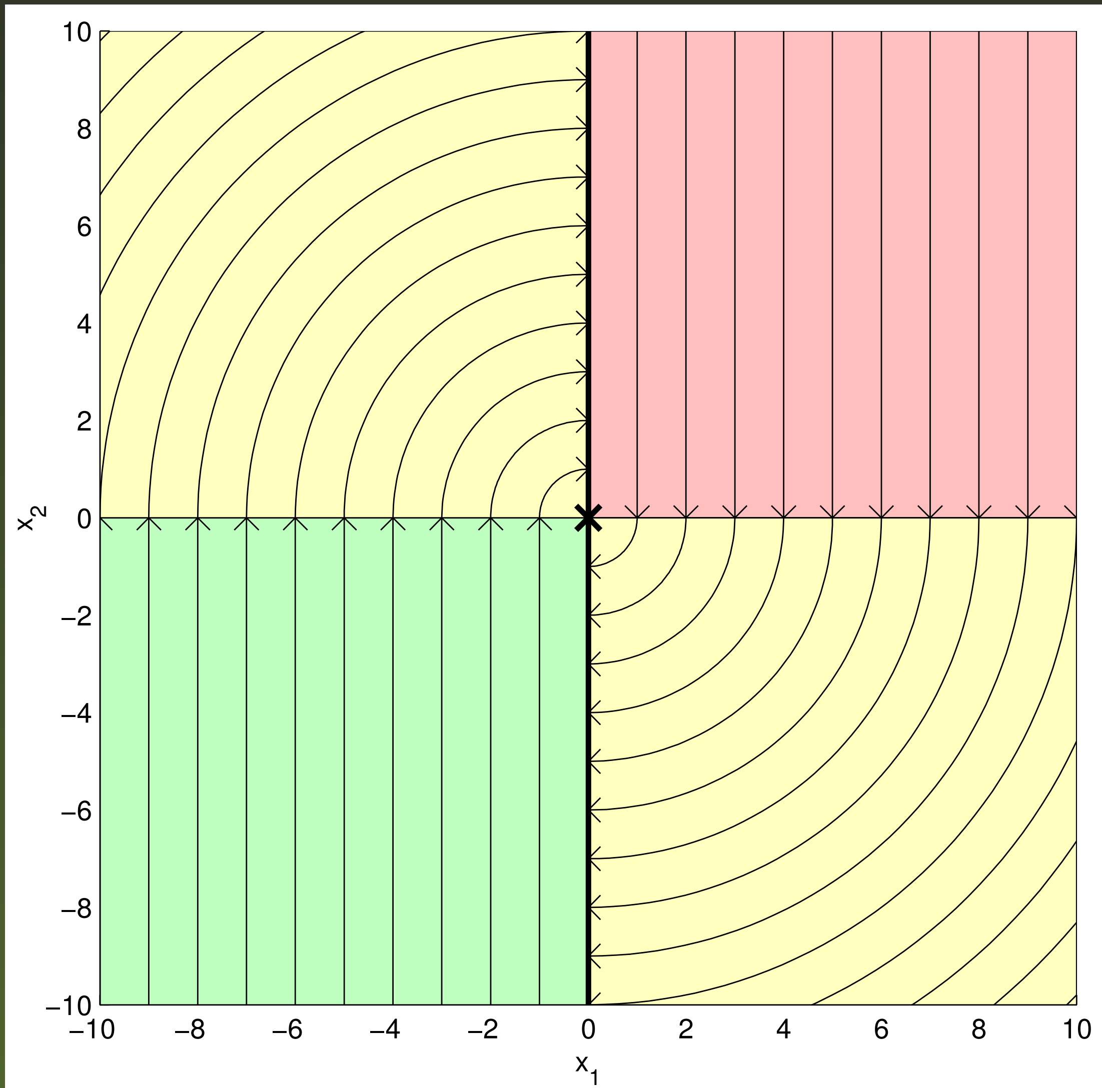
$$\operatorname*{Var}_{[0,\frac{\pi}{2}]} U(\cdot) \to \inf,$$

$$x_1(0-0) = x_1^0, \quad x_2(0-0) = x_2^0, \quad x_1(\tfrac{\pi}{2}+0) = 0, \quad x_2(\tfrac{\pi}{2}+0) = 0.$$

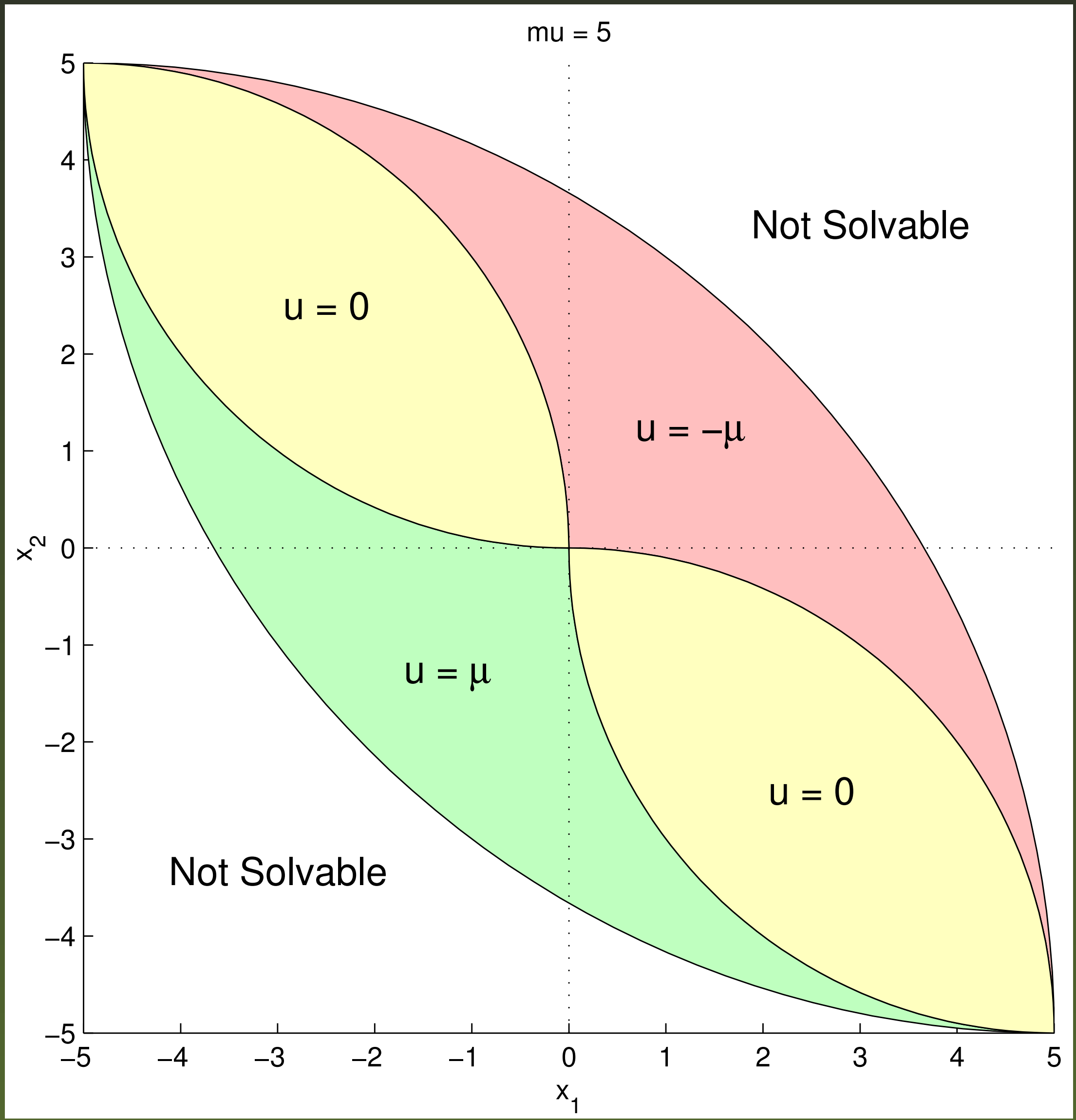
Example: The Ideal Control



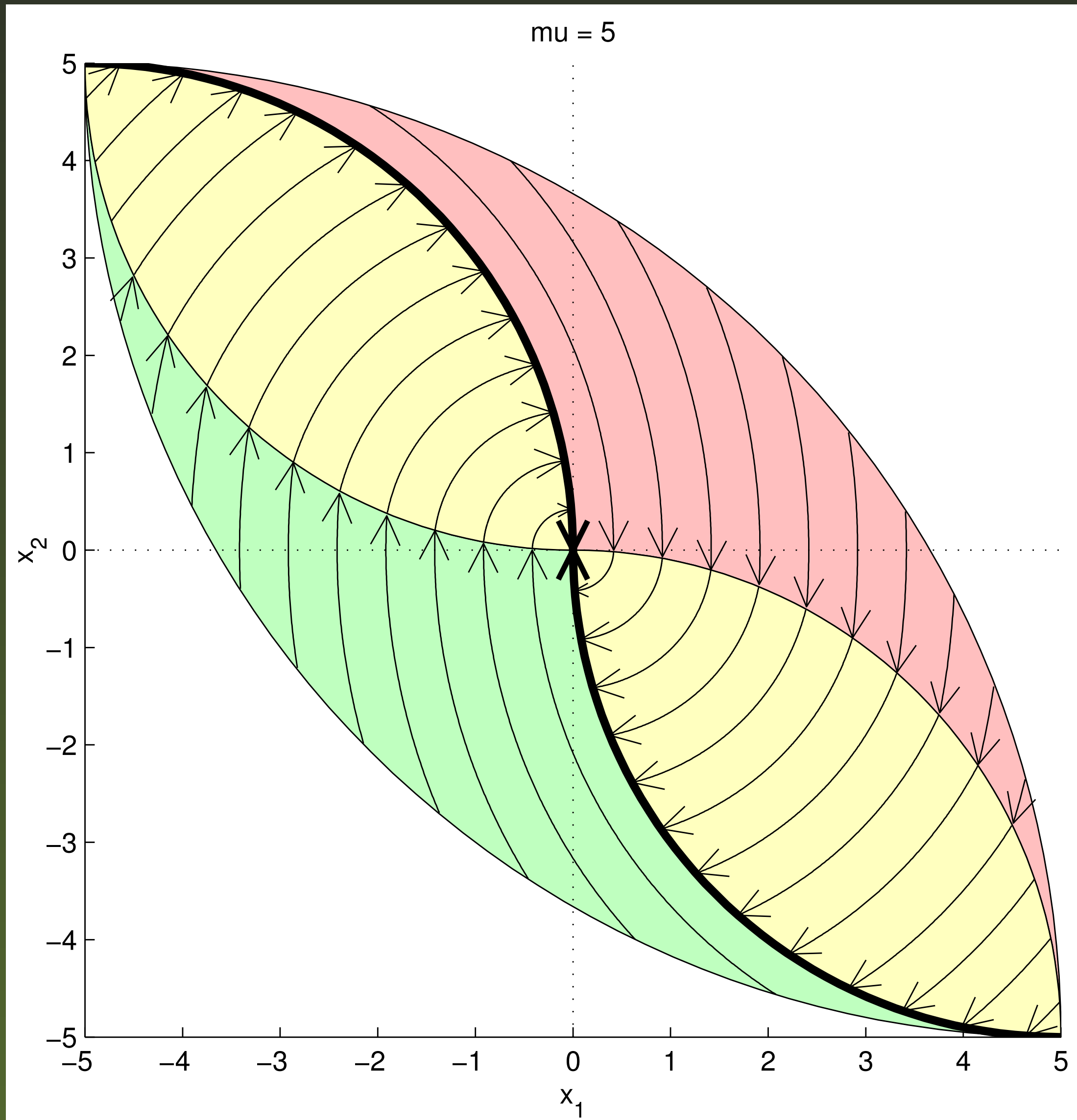
Example: The Ideal Trajectories



Example: The Realistic Control



Example: The Realistic Trajectories



Example: The Convergence