A Dynamic Programming Approach to the Impulse Control Synthesis Problem

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***Abstract*— This paper deals with the problem of synthesizing optimal impulse controls in linear systems through appropriate feedback strategies. Solutions are given within an ideal scheme involving closed-loop delta-function controls, as well as through a "realistic" approximations with delta-type sequences of "ordinary" functions. The solution ia presented through a dynamic programming scheme which indicates related HJB-type variational inequalities for this problem.**

I. Introduction

Solving the problem of control synthesis is one of the main topics in control theory. This may be done within various classes of feedback controls specified in advance. Thus, in the classical theory with hard bounds on the controls, the solutions may turn to be of the "bang-bang" type, so that the synthesized system is described by differential equations with discontinuous right-hand side [1] and switching surfaces [2]–[4].

However, in many applied problems, for example, those related to control in aerospace through instantaneous corrections, control under communication constraints or logically controlled systems the solutions may turn to be of the *impulse* type which requires the control to be of generalized nature, consisting of impulse "delta-functions" or their combination with bang-bang controls or continuous controls. Problems of such type were mostly treated as those of open-loop control (see [4]–[7] etc.), with a well-formalized theory of closed-loop control synthesis still pending.

The present paper indicates the possibility of a dynamic programming approach to problems of impulse control which yields solutions in the form of synthesizing control strategies. The discussion is restricted to linear systems which allows to incorporate both classical theory of distributions and the theory of generalized (viscosity) solutions [8]–[10] to the related variational inequalities of the Hamilton–Jacobi–Bellman (HJB) type (see [11]).

II. The Problem

In this paper we consider a problem of minimizing a generalized Meier–Bolza-type functional along an impulse control system:

$$\begin{cases} J(u(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1+0)) \to \inf, \\ \mathrm{d}x(t) = A(t)x(t)\,\mathrm{d}t + B(t)\,\mathrm{d}U(t), \quad t \in [t_0,t_1], \\ x(t_0-0) = x_0. \end{cases} \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, $U(\cdot) \in BV([t_0,t_1];\mathbb{R}^m)$ is the generalized control, $BV([t_0,t_1];\mathbb{R}^m)$ is the space of m-vector functions of bounded variation. Matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are assumed continuous. The terminal time t_1 is fixed. $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a closed convex terminal function; its presence in the formula for $J(u(\cdot))$ allows to state the principle of optimality.

A particular choice of $\varphi(x) = \mathcal{I}(x \mid \{x_1\})$[1] leads to the well-known problem of steering the controlled system from a point x_0 at time t_0 to a point x_1 at time t_1 with the minimum of variation of the control:

$$\begin{cases} \operatorname*{Var}_{[t_0,t_1]} U(\cdot) \to \inf, \\ \mathrm{d}x(t) = A(t)x(t)\,\mathrm{d}t + B(t)\,\mathrm{d}U(t), \quad t \in [t_0,t_1], \\ x(t_0-0) = x_0, \quad x(t_1+0) = x_1. \end{cases} \tag{2}$$

Problems of such kind have been thoroughly studied (see [4], [5], [12]–[14]). They may be solved using methods of functional analysis and convexity theory. The solution is then an open-loop control. However in the present paper we are interested in a dynamic programming solution which yields a closed-loop control. This paper continues the research of [15].

III. The Ideal Scheme

The *value function* $V(t_0,x_0)$ of problem (1) is the optimal value of $J(U(\cdot))$ given fixed initial position (t_0,x_0). An extended notation $V(t_0,x_0;t_1,\varphi(\cdot))$ will be also used to emphasize the dependence of the optimal value $V(t_0,x_0)$ on terminal time t_1 and terminal function $\varphi(\cdot)$.

Notation $W(t_0,x_0) = W(t_0,x_0;t_1,x_1)$ will be used for the minimal variation of problem (2). As discussed above, $W(t_0,x_0;t_1,x_1) = V(t_0,x_0;t_1,\mathcal{I}(\,\cdot \mid \{x_1\}))$.

We decompose problem (1) into a pair of subproblems:

- find the optimal terminal state x_1 of the trajectory, and
- find the optimal control $U(\cdot)$ in problem (2) under condition $x(t_1+0) = x_1$.

The open-loop solution of the second subproblem is given in [4], [16]. It is summarized in the following statements:

Statement 1: The optimal value in problem (2) may be presented as

$$W(t_0,x_0;t_1,x_1) = \sup_{p\in\mathbb{R}^n} \frac{\langle p, x_1 - X(t_1,t_0)x_0\rangle}{\|B'(\cdot)X'(t_1,\cdot)p\|_{C[t_0,t_1]}}. \tag{3}$$

Here $X(t,\tau)$ is the solution to the matrix differential equation $\partial X/\partial t = A(t)X$, $X(\tau,\tau) = I$.

[1] By $\mathcal{I}(x \mid A)$ we denote the indicator function of the convex set A (zero in A and $+\infty$ outside of A).

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Statement 2: Whenever $W(t_0,x_0)<\infty$, there exists an optimal open-loop control $U(\cdot)$ of a special form

$$\mathrm{d}U(t)=\sum_{i=1}^{n}h_i\delta(t-\tau_i)\,\mathrm{d}t, \tag{4}$$
$$t_0\leqslant\tau_1<\tau_2<\ldots<\tau_n\leqslant t_1,\quad u_i\in\mathbb{R}^m,$$
$$\operatorname*{Var}_{[t_0,t_1]}U(\cdot)=\sum_{i=1}^{n}\|h_i\|=V(t_0,x_0).$$

In other words, if it is possible to steer the controlled system from (t_0,x_0) to (t_1,x_1), then there exist controls for which the optimal value $W(t_0,x_0)$ is attained. Moreover, the optimal value may be attained using only n (or less) jumps, where n is the dimension of the vector x.

Now using (3) the value function of problem (1) may be presented as

$$V(t_0,x_0)=\inf_{x_1\in\mathbb{R}^n}\{\varphi(x_1)+W(t_0,x_0;t_1,x_1)\}. \tag{5}$$

Lemma 1: If $V(t_0,x_0)$ is finite, then there exists an optimal vector x_1^* such that $V(t_0,x_0)=\varphi(x_1^*)+W(t_0,x_0;t_1,x_1^*)$.

Proof: Denote $\Phi(x_1)=\varphi(x_1)+W(t_0,x_0;t_1,x_1)$. From (3) it follows that

$$W(t_0,x_0;t_1,x_1)\geqslant\frac{\|x_1-X(t_1,t_0)x_0\|}{\|B'(t)X'(t_1,\cdot)p_1\|_{C[t_0,t_1]}},$$

where p_1 is a unit vector collinear to $x_1-X(t_1,t_0)x_0$. This means that $W(t_0,x_0;t_1,x_1)\to\infty$ as $\|x_1\|\to\infty$, and the level set

$$L=\{\,x_1\mid\Phi(x_1)\leqslant V(t_0,x_0)+\varepsilon\} \tag{6}$$

is compact. Thus the function $\Phi(x_1)$ attains its minimum value on the set L at some point x_1^*. ∎

A combination of statement 2 and Lemma 1 yields the following result.

Theorem 1: Whenever $V(t_0,x_0)<\infty$ there exists an open-loop control $U(\cdot)$ of type (4) such that $J(U(\cdot))=V(t_0,x_0)$.

We further introduce a semi-norm on $\mathbb{R}^n$

$$\|p\|_{[t_0,t_1]}=\|B'(t)X'(t_1,\cdot)p\|_{C[t_0,t_1]}$$

and define a linear subspace $P_{[t_0,t_1]}=\Big\{p\;\Big|\;\|p\|_{[t_0,t_1]}=0\Big\}$. It has a positive dimension iff the system in (1) is not completely controllable.

For $V(t_0,x_0)$ to be finite it is necessary and sufficient that $\langle p,x_1-X(t_1,t_0)x_0\rangle=0$ when $\|p\|_{[t_0,t_1]}=0$, or, equivalently, $x_1\in X(t_1,t_0)x_0+P^{\perp}_{[t_0,t_1]}$. Since the $\|\cdot\|_{[t_0,t_1]}$ is a norm on $P^{\perp}_{[t_0,t_1]}$, we may rewrite (5) as

$$V(t_0,x_0)=\inf_{x_1\in X(t_1,t_0)x_0+P^{\perp}_{[t_0,t_1]}}\;\sup_{p\in\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}\cap P^{\perp}_{[t_0,t_1]}}\{\varphi(x_1)+\langle p,x_1-X(t_1,t_0)x_0\rangle\}.$$

and refer to the min-max theorem of [17] on changing the order of $\inf$ and $\sup$. This yields

$$V(t_0,x_0)=\sup_{p\in\mathbb{R}^n}\Big[\langle p,X(t_1,t_0)x_0\rangle-\\ -\varphi^*(p)-\mathcal{I}\Big(p\;\Big|\;\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}\Big)\Big], \tag{7}$$

where $\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}$ is the unit ball in the introduced semi-norm, and $\varphi^*(p)$ is the Fenchel conjugate of $\varphi(x)$ [18]. Thus we have proven the following statement:

Theorem 2: The value function $V(t_0,x_0)$ is convex in x and its conjugate is given by

$$V^*(t_0,p)=\varphi^*(X'(t_0,t_1)p)+\\ +\mathcal{I}\Big(X'(t_0,t_1)p\;\Big|\;\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}\Big). \tag{8}$$

Using (8) one may prove the next result:

Theorem 3: The value function $V(t,x;t_1,\varphi(\cdot))$ of problem (1) satisfies the *principle of optimality* in the form of the semigroup property. Namely, for each $\tau\in[t_0,t_1]$

$$V(t_0,x_0;t_1,\varphi(\cdot))=V(t_0,x_0;\tau,V(\tau,\cdot;t_1,\varphi(\cdot))).$$

Note that, unlike problems without impulse controls, in the general case $V(t_1,x;t_1,\varphi(\cdot))\leqslant\varphi(x)$, since from (8) it follows that

$$V^*(t_1,p)=\varphi^*(p)+\mathcal{I}(\,B(t_1)p\mid\mathcal{B}_1).$$

For example, if $\varphi(x)=\mathcal{I}(\,x\mid\{x_1\})$ and $B(t_1)=I$, then $V(t_1,x;t_1,\varphi(\cdot))=\|x-x_1\|\leq\varphi(x)$.

Theorem 4: The value function $V(t,x)$ is the viscosity solution [8] to the Hamilton–Jacobi–Bellman equation:

$$\min\{H_1(t,x,V_t,V_x),\;H_2(t,x,V_t,V_x)\}=0, \tag{9}$$
$$V(t_1,x)=V(t_1,x;t_1,\varphi(\cdot)).$$

$$H_1(t,x,\xi_t,\xi_x)=\xi_t+\langle\xi_x,A(t)x\rangle,$$
$$H_2(t,x,\xi_t,\xi_x)=\min_{u\in S_1}\langle\xi_x,B(t)u\rangle+1.$$

Proof: First estimate the subdifferential $\partial_x^-V(t_0,x_0)$ in x of the value function:

$$\partial_x^-V(t_0,x_0)\subseteq\operatorname{dom}V^*(t_0,\cdot)\subseteq X'(t_1,t_0)\mathcal{B}_{\|\cdot\|_{[t_0,t_1]}}\subseteq\\ X'(t_1,t_0)\mathcal{B}_{\|\cdot\|_{[t_0,t_0]}}=\{\,p\mid\|B'(t_0)p\|\leqslant1\}.$$

which gives $V'(t_0,x_0;0,B(t)u)\geqslant-1$ for $\|u\|\leqslant1$. Here $V'(t_0,x_0;\tau,\xi)$ denotes the directional derivative of $V(t_0,x_0)$ in direction (τ,ξ). If $V(t,x)$ is differentiable at (t_0,x_0), this turns into $H_2(t_0,x_0,V_t,V_x)\geqslant0$.

Then setting in (5) $t_1=t_0+\sigma$, $\varphi(\cdot)=V(t_0+\sigma,\cdot)$ and choosing $x_1=X(t_0+\sigma,t_0)x_0$, we get

$$V(t_0,x_0)\leqslant V(t_0+\sigma,X(t_0+\sigma,t_0)x_0)=w(\sigma).$$

Due to the principle of optimality the function $w(\sigma)$ is non-decreasing in σ, hence its right derivative exists: $w'(\sigma+0)=0$. This yields $V'(t_0,x_0;1,A(t_0)x)\geqslant0$, and if $V(t,x)$ is differentiable at (t_0,x_0), it becomes $H_1(t_0,x_0,V_t,V_x)\geqslant0$.

Finally take an optimal control of the type (4). Then either $\tau_1>t_0$ and $H_1=0$, or $\tau_1=t_0$ and H_2, that is, $\min\{H_1,H_2\}=0$. ∎

Due to (9), in any position (t,x) there are two possibilities for the control. Either $H_1 = 0$, and the control may choose $\mathrm{d}U(t) = 0$, or $H_1 > 0$, in which case it is necessary the $H_2 = 0$, and the control has a jump in direction $-B'(t)V_x$. The magnitude of the jump is to be selected in such a way that after the jump we again have $H_1 = 0$.

However, such reasoning is not yet rigorous enough, since it is still unclear what would a closed-loop system be under such control. A possible way to overcome this difficulty lies in using the extended space-time system [6], [7], [19]:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = A(t(s))x(s) \cdot u^t(s) + B(t(s))u^x(s), \\ \dfrac{\mathrm{d}t}{\mathrm{d}s} = u^t(s). \end{cases} \tag{10}$$

Here s is the parameterizing variable for trajectories of x and t, $s \in [0,S]$, and the right end S is not fixed. The extended control $u(s) = (u^x(s), u^t(s)) \in \mathbb{R}^m \times \mathbb{R}$ is restricted by hard bound $u(s) \in \mathcal{B}_1 \times [0,1]$. The original impulse control problem (1) corresponds to the following problem for system (10):

$$\begin{cases} J(u(\cdot)) = \displaystyle\int_0^S \|u^x(s)\| \,\mathrm{d}s + \varphi(x(S)) \to \inf, \\ t(0) = t_0, \quad t(S) = t_1. \end{cases} \tag{11}$$

It is known [7] that any impulse control and its corresponding state trajectory of the original system (1) may be presented as similar elements of the extended system (10), and that the set of trajectories of (1) is dense in the set of trajectories of (10).

The value function of the problem (11) is the viscosity solution to the the Hamilton–Jacobi–Bellman equation

$$\min_{\substack{u^t \in [0,1] \\ u^x \in \mathcal{B}_1}} H(t,x,V_t,V_x,u^t,u^x) = 0, \tag{12}$$

$$H(t,x,\tau,\xi,u^t,u^x) = \big\{[\tau + \langle \xi, A(t)x \rangle]u^t + \\ + [\langle \xi, B(t)u^x \rangle + \|u^x\|]\big\} = 0,$$

which is equivalent to the HJB equation (9) for the impulse control problem.

Now using (12) it is possible to define control synthesis for (11) as the set of minimizing control vectors in (12):

$$\mathcal{U}^*(t,x) = \bigcup_{(\tau,\xi) \in \partial_C V} \big\{ u \;\big|\; H(t,x,\tau,\xi,u^t,u^x) = 0 \big\}, \tag{13}$$

Here $\partial_C V$ is the Clarke subdifferential [20] of the value function with respect to both variables (t,x).

Since (10) describes all the trajectories of (1), the control (13) may be regarded as a control synthesis for (1).

The closed-loop system under control (13) is a differential inclusion:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x \\ t \end{pmatrix} \in \left\{ \begin{pmatrix} A(t)x & B(t) \\ 1 & 0 \end{pmatrix} u \;\middle|\; u \in \mathcal{U}^*(t,x) \right\}. \tag{14}$$

Since $\mathcal{U}^*(t,x)$ is an upper semicontinuous set-valued function with non-empty compact convex values (this follows from the properties of ∂_C), the solutions to (14) exist and are extendable within the region $(t,x) \in [t_0,t_1] \times \mathbb{R}^n$ (see [1]). Any optimal control and the corresponding state trajectory of (1) satisfies (14). In other words, (14) generates all possible optimal trajectories. However, there still remains an open question whether all trajectories of (14) reaching $t = t_1$ are optimal, which will be the subject of the future work.

Example 1: Consider a one-dimensional impulse control problem of type (2):

$$\begin{cases} \operatorname*{Var}_{[-1,1]} u(\cdot) \to \inf, \quad \mathrm{d}x = B(t)\,\mathrm{d}U(t),\ t \in [-1,1], \\ x(-1-0) = x_0,\ x(1+0) = 0. \end{cases}$$

We shall present the exact formulas for the value function (5) and for the control synthesis (13) for certain choices of $B(t)$. To shorten the expressions we use the following abbreviations for the control vectors $u = (u^x, u^t)$: $\circ = (0,0)$, $\uparrow = (1,0)$, $\downarrow = (-1,0)$, $\rightarrow = (0,1)$.

- For $B(t) = 1 - t$, $V(t,x)\,|x|/(1-t)$ and
$$\mathcal{U}^*(t,x) = \operatorname{conv}\begin{cases} \{\circ,\downarrow\}, & x > 0, \\ \{\circ,\uparrow\}, & x < 0, \\ \{\circ,\downarrow,\uparrow,\rightarrow\}, & x = 0. \end{cases}$$
Since $B(t)$ is decreasing, the control should jump to zero as soon as possible.
- For $B(t) = 1 + t$, $V(t,x) = |x|/2$ and
$$\mathcal{U}^*(t,x) = \operatorname{conv}\begin{cases} \{\circ,\rightarrow\}, & t < 1, \\ \{\circ,\rightarrow,\downarrow\}, & t = 1, x > 0, \\ \{\circ,\rightarrow,\uparrow\}, & t = 1, x < 0, \\ \{\circ,\downarrow,\uparrow,\rightarrow\}, & t = 1, x = 0. \end{cases}$$
Here $B(t)$ is increasing and the control should wait until the final instant of time to make a jump.
- For $B(t) = 1 - t^2$ one has $V(t,x) = |x|$, $t \leqslant 0$ and $V(t,x) = |x|/(1-t^2)$, $t \geqslant 0$.
$$\mathcal{U}^*(t,x) = \operatorname{conv}\begin{cases} \{\circ,\rightarrow\}, & t < 0, \\ \{\circ,\rightarrow,\downarrow\}, & t = 0, x > 0, \\ \{\circ,\rightarrow,\uparrow\}, & t = 0, x < 0, \\ \{\circ,\downarrow,\uparrow,\rightarrow\}, & t \geqslant 0, x = 0, \\ \{\circ,\downarrow\}, & t > 0, x > 0, \\ \{\circ,\uparrow\}, & t > 0, x < 0. \end{cases}$$
When $t < 0$, the control should wait for a jump at time $t = 0$, when $B(t)$ is at maximum. When $t \geqslant 0$, the control should jump immediately, since further on $B(t)$ will only decrease.

IV. The Double Constraint Approach

In the previous section an ideal scheme has been considered where solutions to the control problem are generalized functions. Here we shall present a “realistic” approach in which controls are “ordinary” bounded functions, though their bound may be arbitrarily large or even tend to infinity.

Let us introduce an additional hard bound on the control in (1), $u(t) \in \mathcal{B}_\mu$, and consider the corresponding problem:

$$\begin{cases} J(u(\cdot)) = \displaystyle\int_{t_0}^{t_1} \|u(t)\| \,\mathrm{d}t + \varphi(x(t_1)) \to \inf, \\ \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in [t_0,t_1], \\ \quad x(t_0) = x_0, \quad \|u(t)\| \leqslant \mu. \end{cases} \tag{15}$$

Remark 1: The solution of problem (15) exists due to the theorem of Weierstrass: the set of admissible controls is weakly compact (since it is bounded, closed and convex in $L_2([t_0,t_1];\mathbb{R}^m)$), and the objective function $J(u(\cdot))$ is weakly lower semicontinuous (because it is convex and lower semicontinuous in $L_2([t_0,t_1];\mathbb{R}^m)$).

The value function $V_\mu(t_0,x_0)=V_\mu(t_0,x_0;t_1,\varphi(\cdot))$ of this problem is the viscosity solution [8] to the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V_\mu}{\partial t}+\min_{u\in\mu\mathcal{B}_1}\left\{\left\langle\frac{\partial V_\mu}{\partial x},A(t)x(t)+B(t)u\right\rangle+\|u\|\right\}=0 \tag{16}$$

with initial condition $V_\mu(t_1,x)=\varphi(x)$. Thus, except for some degenerate cases the control values are only from $S_\mu\cup\{0\}$.

The solution of (15) may be presented as an optimal value in a finite-dimensional optimization problem:

$$\begin{aligned}V_\mu(t,x)=\sup_{p\in\mathbb{R}^n}\Big\{&\langle p,X(t_1,t_0)x_0\rangle-\\ &-\mu\int_t^{t_1}\left(\|B'(\tau)X'(t_1,\tau)p\|-1\right)_+\mathrm{d}\tau-\varphi^*(p)\Big\},\end{aligned} \tag{17}$$

and its conjugate function in x is given by

$$V_\mu^*(t,p)=\varphi^*(p)+\mu\int_t^{t_1}\left(\|B'(\tau)X'(t_1,\tau)p\|-1\right)_+\mathrm{d}\tau. \tag{18}$$

Here $a_+=\max\{a,0\}$.

Note that as μ tends to infinity, the expressions (17), (18) turn into (7) and (8) respectively. For the case $x\in\mathbb{R}^1$ it may further be shown that for each position (t,x) there exists a constant $C>0$ such that

$$0\leqslant V(t,x)-V_\mu(t,x)=C\mu^{-1}$$

The HJB equation (9) may be also formally derived through a limit transition from (16) as $\mu\to\infty$.

The optimal feedback control strategy is the minimizer in (16), and at points of differentiability of $V_\mu(t,x)$ it may be written as follows:

$$\mathcal{U}_\mu^*(t,x)=\begin{cases}\{0\}, & \|\zeta\|<1;\\ [0,-\mu\zeta], & \|\zeta\|=1;\\ \left\{-\mu\dfrac{\zeta}{\|\zeta\|}\right\}, & \|\zeta\|>1,\end{cases} \tag{19}$$

where $\zeta=B'(t)\frac{\partial V_\mu}{\partial x}$. The strategy (19) satisfies the conditions of existence and extendability of trajectories of the closed-loop system in the form of the differential inclusion [1]:

$$\dot{x}(t)\in A(t)x(t)+B(t)\mathcal{U}_\mu^*(t,x). \tag{20}$$

However, in (19) it is not possible to proceed to the limit as in (17) and (16). In particular, it is not clear what the closed-loop system for the problem (1) will look like. Another problem is that if the closed-loop (20) were implemented using some discretization technique (e.g. Euler' scheme with time step σ), one should choose $\sigma=O(\mu^{-1})$ in order to attain admissible approximation accuracy, which may be unfeasible for large values of μ. To avoid this difficulties, we introduce the following definition of control synthesis for this problem.

Definition 1: The pair of functions $\mathfrak{U}=\{u(t,x;\mu),\,\theta(t,x;\mu)\}$ ("magnitude" and "duration"), such that

$$\begin{aligned}&u(t,x;\mu)\in S_1\cup\{0\}, && u(t,x;\mu)\underset{\mu\to\infty}{\to}u_\infty(t,x),\\ &\theta(t,x;\mu)\geqslant 0, && \mu\theta(t,x;\mu)\underset{\mu\to\infty}{\to}m_\infty(t,x),\end{aligned}$$

is called the *feedback control strategy for* (1).

The components u and θ of such feedback strategy resemble the components u^x and u^t in the ideal feedback control (13). The component $u(t,x)$ is the direction of the control impulse which is issued on interval $[t,t+\theta(t,x)]$. Note that as $\mu\to\infty$, $\theta\to 0$ and in the limit one has a delta-function as control.

Definition 2: Fix a control strategy $\mathfrak{U}$, number $\mu>0$ and a partition $t_0=\tau_0<\tau_1<\ldots<\tau_s=t_1$ of interval $[t_0,t_1]$. An *approximating motion* of system (1) is the solution to the differential equation

$$\begin{aligned}&\tau_i^*=\tau_i\wedge\theta(\tau_{i-1},x_\Delta(\tau_{i-1});\mu),\\ &\dot{x}_\Delta(\tau)=\mu B(t)u(\tau_{i-1},x_\Delta(\tau_{i-1});\mu),\quad \tau_{i-1}<\tau<\tau_i^*,\\ &x_\Delta(\tau_i)=x_\Delta(\tau_i^*).\end{aligned}$$

Number $\sigma=\max\{\tau_i-\tau_{i-1}\}$ is the *diameter* of the partition.

Definition 3: A *constructive motion* of system (1) under feedback control $\mathfrak{U}$ is a piecewise continuous function $x(t)$, which is the pointwise limit of approximating motions $x_\Delta(t)$ as $\mu\to\infty$ and $\sigma\to 0$.

Suppose that current position of (1) is $(\bar{t},\bar{x})$, and from (9) it follows that control has a jump $\bar{h}\delta(t-\bar{t})$. Then the corresponding feedback strategy values $\bar{u}=u(\bar{t},\bar{x};\mu)$, $\bar{\theta}=\theta(\bar{t},\bar{x};\mu)$ are to be chosen in such a way that the following equality would hold:

$$B(t)\bar{h}=\mu\left[\int_{\bar{t}}^{\bar{t}+\bar{\theta}}B(t)\,\mathrm{d}t\right]\bar{u}.$$

In the limit this yields

$$u(\bar{t},\bar{x};\mu)\underset{\mu\to\infty}{=}\frac{\bar{h}}{\|\bar{h}\|},\qquad \mu\theta(\bar{t},\bar{x};\mu)\underset{\mu\to\infty}{=}\|\bar{h}\|.$$

That is, an impulse $\bar{h}\delta(t-\bar{t})$ is (approximately) replaced by a platform of magnitude μ, with direction $\bar{h}$ and duration $\mu^{-1}\|\bar{h}\|$.

V. The Two-dimensional Case

In this section special case of (2) is studied, namely the two-dimensional stationary system with a scalar control:

$$\begin{cases}\mathrm{d}x(t)=Ax(t)\,\mathrm{d}t+b\,\mathrm{d}U(t),\quad t\in[t_0,t_1],\\ x(t_0-0)=x_0,\quad x(t_1+0)=x_1,\\ x(t)\in\mathbb{R}^2,\quad b\in\mathbb{R}^2,\quad U(t)\in\mathbb{R}^1.\end{cases} \tag{21}$$

For such systems the time-optimal and fuel-optimal control problems are considered. It is possible to construct an explicit form of optimal control and to study the so-called *IG (Impulse-Generating) surfaces* (the analogues of switching surfaces in bang-bang control) on which optimal control has a jump and controlled trajectory is discontinuous. These results are based on the geometry of reachability sets and on the related theorem on impulse control structure.

The following theorem shows the structure of forward and backward reachability sets for the controlled system (21):

Theorem 5: Let $\mathcal{X}_\mu[t;t_0,\mathcal{X}_0]$ be the reachability set for system (21) and $\mathcal{W}_\mu[t;t_1,\mathcal{X}_1]$ be the backward reachability set for this system under the condition $\operatorname{Var}_{[t_0,t_1]} U(\cdot) \leqslant \mu$. Then these sets may be presented in the following form:

$$\mathcal{X}_\mu[t;t_0,\mathcal{X}_0] = \mathrm{e}^{A(t-t_0)}\mathcal{X}_0 + \mu \operatorname{conv} \bigcup_{\tau\in[t_0,t]} \mathrm{e}^{A(t-\tau)}[-b,b],$$

$$\mathcal{W}_\mu[t;t_1,\mathcal{X}_1] = \mathrm{e}^{A(t-t_1)}\mathcal{X}_1 + \mu \operatorname{conv} \bigcup_{\tau\in[t,t_1]} \mathrm{e}^{A(t-\tau)}[-b,b].$$

Consider $x_0 \in \mathcal{W}_\mu[t_0;t_1,\{x_1\}]$. Then, due to the statement 4 there exists an impulse control of type (4) with at most two impulses, i.e. for some $s \in [t_0,t_1)$

$$U(t) = h_1\chi(t-s) + h_2\chi(t-t_1),\ h_1,h_2 \in \mathbb{R}^1, \qquad (22)$$

which translates system (21) from the position (t_0-0,x_0) to the position (t_1+0,x_1), and

$$\operatorname*{Var}_{[t_0,t_1]} U(\cdot) = \sum_{i=1}^{2}|h_i| \leqslant \mu.$$

Consider the time-optimal control problem for system (21), looking for $t_1 - t_0 \to \min$. Without any loss of generality one may fix $x_1 = 0$. If it is possible to steer this system from the state x_0 to the origin, it is also possible to do this only with two impulses. Thus the optimal control can be presented as in (22).

We will study a special case when $s = t_0$, i.e. when the optimal control has an impulse at the starting time. The set of such states x^0 will be referred to as the *IG-set* $\mathcal{J}_t[\mu]$. It is convenient to represent $\mathcal{J}_t[\mu]$ as a union of sets of equal impulses $\{r(h_1,h_2)\}$. By definition, if $x^0 \in r(h_1,h_2)$, then the optimal control from x^0 to the origin has impulses h_1 at the starting instant of time and h_2 at the final instant. So that if $T_0(x^0)$ is the optimal time to steer the system (21) from the state x^0 to the origin, then the set of equal impulses is

$$r(h_1,h_2) = \left\{ x \in \mathbb{R}^n \;\middle|\; x + h_1 b + h_2 \mathrm{e}^{-T_0(x)A} b = 0 \right\}$$

Lemma 2: Divide all systems (21) into two classes:

1) matrix A has real eigenvalues of different sign, and
2) all other systems.

In the first case $r(h_1,h_2) = \left\{ -h_1 b - h_2 \mathrm{e}^{-\tau A} b \;\middle|\; \tau \in [0,t^*] \right\}$. In the second case

$$r(h_1,h_2) = \begin{cases} \left\{ -h_1 b - h_2 \mathrm{e}^{-\tau A} b \;\middle|\; \tau \in [0,t^*] \right\},\ h_1h_2 < 0; \\ \varnothing,\ h_1h_2 \geqslant 0. \end{cases}$$

Here $t^* = \min\{t_a,t_b\}$, and

$$t_a = \inf\left\{ t \;\middle|\; e^{-tA}b \in \operatorname{int}\mathcal{W}_\mu[0;t,\{0\}] \right\},$$
$$t_b = \inf\left\{ t \mid b \in \operatorname{int}\mathcal{W}_\mu[0;t,\{0\}] \right\}.$$

The following theorem describes the structure of the IG-set:

Theorem 6: The IG-set $\mathcal{J}_t[\mu]$ is the union of equal impulses sets

$$\mathcal{J}_t[\mu] = \bigcup_{|h_1|+|h_2|=\mu,\,h_1\neq 0} r(h_1,h_2)$$

After calculating the IG-set the optimal control synthesis is expressed in the following form. When $x_0 \in r(h_1,h_2)$, for some h_1 and h_2 such that $|h_1|+|h_2| = \mu$, then there is a jump $h_1\delta(t-t_0)$. Otherwise, that is when $x_0 \notin \mathcal{J}_t[\mu]$, there is no jump at time t_0.

A similar result may be established for the energy-optimal control problem. It simply follows from the theorem about IG-set structure in case of the time-optimal problem:

Theorem 7: An IG-set $\mathcal{J}_\mu[t]$ has the following representation:

- if $t > t^*$ then $\mathcal{J}_\mu[t] = \varnothing$;
- If $t \leqslant t^*$ then $\mathcal{J}_\mu[t] = \left\{ -h_1 b - h_2 \mathrm{e}^{-tA} b \right\}$ where h_1 and h_2 must have the same sign for case 2) of Lemma 2 and arbitrary signs for case 1).

The time parameter t^* may be calculated directly from system (21).

The control synthesis strategy is also similar to the one for the time-optimal system.

Example 2: Consider the following problem of stopping a pendulum by impulse controls:

$$\begin{cases} \operatorname*{Var}_{[0,\frac{\pi}{2}]} U(\cdot) \to \inf, \\ \begin{cases} \mathrm{d}x_1(t) = x_2(t)\,\mathrm{d}t, \\ \mathrm{d}x_2(t) = -x_1(t)\,\mathrm{d}t + \mathrm{d}U(t), \end{cases} \quad 0 \leqslant t \leqslant \frac{\pi}{2}, \\ x_1(0-0) = x_1^0, \quad x_2(0-0) = x_2^0, \\ x_1(\frac{\pi}{2}+0) = 0, \quad x_2(\frac{\pi}{2}+0) = 0. \end{cases}$$

The exact solution of this problem is as follows. If at current position (t,x) one has

$$t > -\arcsin((x_2 \operatorname{sign} x_1)(x_1^2+x_2^2)^{-1/2}),$$

then optimal control has a jump of an amplitude h_1 that solves

$$t = -\arcsin((x_2+h_1)\operatorname{sign} x_1(x_1^2+(x_2+h_1)^2)^{-1/2}).$$

Otherwise the control should wait until $x_1 = 0$ to have a jump with amplitude $h_2 = -x_2$ straight to the origin. The optimal trajectories that start at $t = 0$ are shown in Fig. 1.

The corresponding double-constraint control synthesis defined by (19) (for $t = 0$) is shown in Fig. 2. Note that the state space is divided into four domains: three domains R_0, $R_{-\mu}$, R_μ correspond to control values 0, μ, $-\mu$ and an outer domain $R_\varnothing$ contains starting positions from which it is not possible to attain the origin (the problem is not solvable).

As $\mu \to \infty$, the domain R_0 fills the second and fourth quadrants, the domains $R_{-\mu}$ and R_μ are to fill the first and third quadrant respectively (Fig. 3). This exactly corresponds to the ideal impulse control presented in Fig. 1.

VI. Conclusion

This paper presents a Dynamic Programming theory for *closed-loop impulse control* in systems with original linear structure through equations or variational inequalities of the HJB type. The suggested approach allows propagation to impulse control problems which involve derivatives of delta-functions along the lines of [16].

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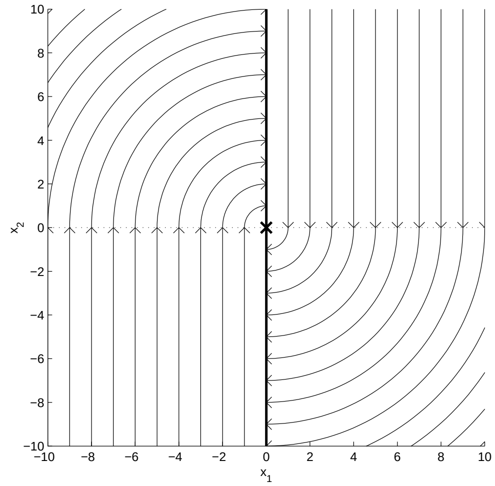

Fig. 1. Ideal control trajectories

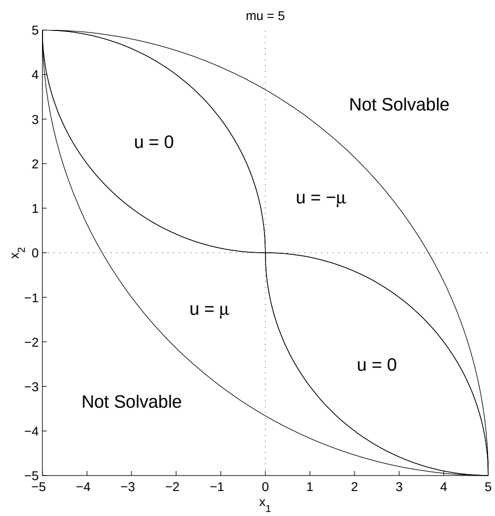

Fig. 2. Double-constraint control trajectories

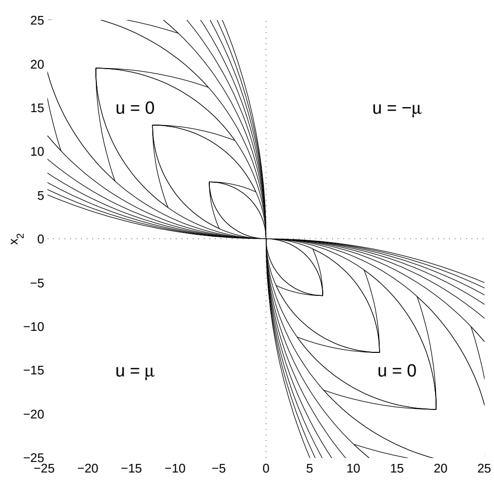

Fig. 3. The convergence of double-constraint controls to the impulse control