On the Theory of Fast Controls under Disturbances ⋆

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Abstract: Fast controls are those that act within a very small time horizon. They are treated as bounded approximations of generalized impulse controls (belonging to the class of higher-order distributions). In this paper, we investigate the fast controls in feedback form under unknown-but-bounded disturbances.

Keywords: impulse control, feedback, fast controls, set-membership uncertainty

1. INTRODUCTION

In this paper we consider the problem of synthesizing fast controls under unknown-but-bounded disturbances.

The term "fast control" stands for bounded approximations of generalized control inputs (Kurzhanski and Daryin (2008)) in the form of higher-order distributions (see generalized functions in Gelfand and Shilov (1964); Schwartz (1950)).

In its turn, a problem with generalized control inputs may be reduced to an "ordinary" impulse control problem using impulses of lowest order (Kurzhanski and Osipov (1969); Daryin and Kurzhanski (2008)). Such problems were introduced and studied in an open-loop form by Krasovski (1957); Neustadt (1964). However, due to the presence of disturbances, it is necessary to develop closed-loop solutions. Here we do this along the lines of paper by Kurzhanski (1999), with bounded controls replaced by those of impulsive type.

The solution of the problem considered here comes in the following four steps:

(1) state the problem with generalized control inputs;
(2) reduce it to an "ordinary" impulse control problem;
(3) solve the last control problem in the class of closed-loop controls;
(4) approximate the solutions by realistic fast controls.

Note that the third step in general involves the solution of a variational inequality of Hamilton–Jacobi–Bellman–Isaacs type. However, in the case of one-dimensional state space it is possible to get an explicit representation for the value function, which will be demonstrated in Section 4.

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2. GENERALIZED CONTROL PROBLEM UNDER DISTURBANCES

2.1 Problem Statement

Consider system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)v(t). \tag{1}$$

Here $u(t) \in \mathbb{R}^m$ is a *generalized control* in the sense defined below. Matrix function $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are taken to be k times differentiable on the interval $\alpha \le t \le \beta$.

The disturbance $v(t)$ is bounded: it is a piecewise-continuous function, taking values in a given non-empty convex compact set $\mathscr{Q}(t)$.

The generalized control $u(t)$ is chosen from the space $D^*_{k,m}[\alpha,\beta]$ of linear functionals on normed linear space $D_{k,m}[\alpha,\beta]$ (Gelfand and Shilov (1964); Schwartz (1950)). The latter consists of k times differentiable functions $\varphi(t) : [\alpha,\beta] \to \mathbb{R}^m$ with support set contained in (α,β), endowed with a norm $\mathscr{G}[\varphi]$.

The norm $\mathscr{G}[\varphi]$ defines a conjugate norm $\mathscr{G}^*[u]$ in space $D^*_{k,m}[\alpha,\beta]$. The control is then a distribution of order $k_u \le k$, and the trajectories of (1) are distributions from $D^*_{k-1,n}[\alpha,\beta]$.

Let $f^{(\alpha)}$ and $f^{(\beta)}$ be two distributions from $D^*_{k,n}[\alpha,\beta]$ concentrated at points t_α and t_β respectively. We call $f^{(\alpha)}$ the *initial*, and $f^{(\beta)}$ the *terminal* distribution.

Given the realization of a piecewise-continuous disturbance $v(t)$, an *admissible control* $u(t)$ is a distribution from $D^*_{k,m}[\alpha,\beta]$, contained within the interval $[t_\alpha, t_\beta]$, where $\alpha < t_\alpha \le t_\beta < \beta$, ensuring the existence of distribution $x(t) \in D^*_{k-1,n}[\alpha,\beta]$ which solves equation

$$\dot{x}(t) = A(t)x + B(t)u + C(t)v(t) + f^{(\alpha)} - f^{(\beta)}. \tag{2}$$

Problem 1. For a given distribution $f^{(\alpha)}$ and a fixed time interval $[t_\alpha, t_\beta]$, find a closed-loop control strategy which generates admissible controls $u(t) \in D^*_{k,m}[\alpha,\beta]$, minimizing the functional

$$J(u, f^{(\beta)}) = \max\left\{ \mathscr{G}^*[u] + \varphi(f^{(\beta)}) \;\middle|\; v(\cdot) \in \mathscr{Q}(\cdot) \right\}. \tag{3}$$

Here $\mathscr{G}^*[u]$ is a conjugate norm in the space $D^*_{k,m}[\alpha,\beta]$ defined by the norm $\mathscr{G}[\varphi]$, and $\varphi(f)$ is a proper convex closed function, bounded from below.

2.2 Reduction to the "Ordinary" Impulse Control Problem

In order to properly define a closed-loop control strategy for Problem 1, we first reduce the latter to an ordinary (first-order) impulse control problem, following Kurzhanski and Osipov (1969).

Define functions $L_j(t)$ by recurrent relations

$$L_0(t) = B(t), \quad L_j(t) = A(t)L_{j-1}(t) - L'_{j-1}(t), \qquad (4)$$

$j = \overline{1,k}$, and form a matrix $\mathscr{B}(t) = (L_0(t)\ L_1(t)\ \cdots\ L_k(t))$.

The controls $U(t) = \left(U_0^T(t)\ U_1^T(t)\ \cdots\ U_k^T(t)\right)^T$ are chosen from the class $BV([t_\alpha,t_\beta];\mathbb{R}^{m(k+1)})$ of functions of bounded variation (each of functions $U_j(t)$, $j=0,\dots,k$ is with values in $\mathbb{R}^m$). Then the corresponding control in Problem 1 is

$$u(t) = \sum_{j=0}^{k}(-1)^j \frac{d^{j+1}U_j}{dt^{j+1}}.$$

Finally we define the end point of the trajectory $x^\alpha = x(t_\alpha) = \sum_{j=0}^k L_{A,j}(t_\alpha)\alpha_j$ and a terminal functional $\Phi(x) = \min\left\{\varphi(f^{(\beta)}) \,\middle|\, \sum_{j=0}^k L_{A,j}(t_\beta)\beta_j = x\right\}$, where $L_{A,j}(t)$ is defined by recurrence relations similar to (4) but with initial condition $L_{A,0}(t) = I$.

Problem 2. Find a closed-loop control that generates admissible control trajectories $U(\cdot) \in BV([t_\alpha,t_\beta],\mathbb{R}^{m(k+1)})$ minimizing the functional

$$J(U(\cdot)) = \max_{v(\cdot)\in\mathscr{Q}(\cdot)}\{\mathrm{Var}\{U(\cdot)\mid[t_\alpha,t_\beta]\} + \Phi(x(t_\beta+0))\}$$

along the trajectories of

$$dx(t) = A(t)x(t)dt + \mathscr{B}(t)dU(t) + C(t)v(t), \quad x(t_\alpha) = x^\alpha.$$

This problem is treated in the next section.

3. IMPULSE FEEDBACK CONTROL UNDER DISTURBANCES

3.1 Problem Formulation

Consider a linear system

$$dx(t) = A(t)x(t)dt + B(t)dU(t) + C(t)v(t)dt. \qquad (5)$$

Here $t\in[t_0,t_1]$, $x(t)\in\mathbb{R}^n$ is the state vector, $U(t)$ is a generalized control. $U(\cdot)$ belongs to the space $BV([t_0,t_1])$ of functions of bounded variation. We also have the unknown disturbance input $v(t)\in\mathbb{R}$ with values restricted to a closed compact set $\mathscr{Q}(t)$. Given are matrix functions $A(t)\in\mathbb{R}^{n\times n}$, $B(t)\in\mathbb{R}^n$, $C(t)\in\mathbb{R}^n$ that are continuous.

The problem is to minimize the functional

$$\max_{v(\cdot)\in\mathscr{Q}(\cdot)}\{\mathrm{Var}\,U(\cdot) + \varphi(x(t_1))\} \to \inf.$$

The initial condition $x(t_0) = x_0$ is given and initial time t_0 and terminal time t_1 are fixed.

The problem will be solved using Hamiltonian techniques in the form of Dynamic Programming. In order to introduce the value function for closed-loop controls, we shall follow the scheme described by Kurzhanski (1999).

3.2 Min-Max Value Function

The min-max value function is defined as

$$V^-(t_0,x_0) = \min_{U(\cdot)}\max_{v(\cdot)}[\mathrm{Var}\,U(\cdot)+\varphi(x(t_1+0)) \mid x(t_0)=x_0].$$

Here $x(t)$ is the trajectory of system (5) corresponding to a fixed control $U(\cdot)$ and disturbance $v(\cdot)$.

The function V^- may be calculated as follows. First, we take the maximum over $v(\cdot)$. Note that $\mathrm{Var}\,U(\cdot)$ does not depend on $v(\cdot)$, and the right end of the trajectory $x(t_1+0)$ may be expressed as

$$x(t_1+0) = \underbrace{X(t_1,t_0)x_0 + \int_{t_0}^{t_1+0} X(t_1,t)B(t)dU(t)}_{\hat{x}(t_1+0)} + \underbrace{\int_{t_0}^{t_1} X(t_1,t)C(t)v(t)dt}_{\mathbf{v}(t_1)} = \hat{x}(t_1+0) + \mathbf{v}(t_1).$$

Here $X(t,\tau)$ is the solution to the following linear matrix ODE: $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.

The vector $\hat{x}(t_1+0)$ is the right end of the trajectory with no disturbance. The vector $\mathbf{v}(t_1)$ belongs to the set

$$\mathbf{Q} = \int_{t_0}^{t_1} X(t_1,t)C(t)\mathscr{Q}(t)dt.$$

Now, employing convex analysis (Rockafellar (1999)) we first get

$$\max_{v(\cdot)\in\mathscr{Q}(\cdot)}\varphi(x(t_1+0)) = \max_{\mathbf{v}\in\mathbf{Q}}\varphi(\hat{x}(t_1+0)+\mathbf{v}) =$$
$$\max_{\mathbf{v}\in\mathbf{Q}}\max_{p\in\mathbb{R}^n}\{\langle\hat{x}(t_1+0)+\mathbf{v},p\rangle - \varphi^*(p)\} =$$
$$\max_{p\in\mathbb{R}^n}\{\langle\hat{x}(t_1+0),p\rangle + \rho\,(p\mid\mathbf{Q}) - \varphi^*(p)\} = \psi(\hat{x}(t_1+0)),$$

where $\psi(\hat{x}(t_1+0))$ is a convex function whose conjugate is $\psi^*(p) = \mathrm{conv}\,\{\varphi^*(p) - \rho\,(p\mid\mathbf{Q})\}$.

Secondly, we calculate the minimum over $U(\cdot)$. Since now

$$V^-(t_0,x_0) = \min_{U(\cdot)}[\mathrm{Var}\,U(\cdot) + \psi(\hat{x}(t_1+0)) \mid x(t_0) = x_0],$$

Hence this is an impulse control problem without disturbance. The value function is

$$V(t_0,x_0) =$$
$$= \max_{p\in\mathbb{R}^n}\left\{\left\langle X^T(t_1,t_0)p,x\right\rangle - \psi^*(p) - \mathscr{I}(p\mid\mathscr{B}_V[t_0,t_1])\right\},$$

(see Kurzhanski and Daryin (2008)), where

$$\mathscr{B}_V[t_0,t_1] = \{p \mid \|p\|_V \le 1\},$$
$$\|p\|_V = \max\{\|B^T(\tau)X^T(t_1,\tau)p\| \mid \tau\in[t_0,t_1]\},$$

where $\|\ell\|$ is the Euclidean norm. $\mathscr{B}_V[t_0,t_1]$ is a unit ball in $\mathbb{R}^n$ whose defined for the the interval $[t_0,t_1]$.

3.3 Value Function with Corrections

For the min-max value function calculated in the previous section, we shall use an extended notation $V^-(t_0,x_0) = V^-(t_0,x_0;t_1,\varphi(\cdot))$.

Let $t_0 = \tau_N < \tau_{N-1} < \cdots < \tau_1 < \tau_0 = t_1$ be some partition of the interval $[t_0,t_1]$. It will be denoted by $\mathscr{T}$, and $\mathrm{diam}\,\mathscr{T}$ is $\max\{\tau_k - \tau_{k+1}\}$.

Define the value function with corrections $V^-_{\mathscr{T}}(t,x)$ by the following recurrent relations:

$$V^-_{\mathscr{T}}(\tau_0, x) = V^-(t_1, x; t_1, \varphi(\cdot));$$
$$V^-_{\mathscr{T}}(\tau_{k+1}, x) = V^-(\tau_{k+1}, x; \tau_k, V^-_{\mathscr{T}}(\tau_k, x)).$$

Function $V^-_{\mathscr{T}}(t,x)$ may be interpreted as the value function for the sequential min-max problem, when at instants τ_k the control obtains information on the current state $x(t)$.

Note that if $\mathscr{T}'$ is a subpartition of $\mathscr{T}$, then clearly $V^-_{\mathscr{T}'}(t,x) \le V^-_{\mathscr{T}}(t,x)$.

3.4 Closed-Loop Value Function

Denote

$$\mathscr{V}^-(t,x) = \inf_{\mathscr{T}} V^-_{\mathscr{T}}(t,x).$$

It may be proved (similar to Kurzhanski and Daryin (2008)) that the value function $\mathscr{V}^-(t,x)$ satisfies a Hamilton–Jacobi–Bellman–Isaacs type variational inequality:

$$\min\{\mathscr{H}_1, \mathscr{H}_2\} = 0,$$
$$\mathscr{H}_1(t,x) = \mathscr{V}^-_t + \max_{v\in\mathscr{Q}} \left\langle \mathscr{V}^-_x, A(t)x + C(t)v \right\rangle,$$
$$\mathscr{H}_2(t,x) = \min_{\|h\|=1} \{\|h\| + \left\langle \mathscr{V}^-_x, B(t)h \right\rangle\},$$
$$\mathscr{V}^-(t_1,x) = \mathscr{V}^-(t_1, x; t_1, \varphi(\cdot)).$$

Here the Hamiltonian $\mathscr{H}_1$ corresponds to the motion without control ($dU = 0$), and $\mathscr{H}_2$ corresponds to the jumps generated by control impulses. Therefore, the last variational inequality may be interpreted as follows: if for $x(\tau)$ we have $\mathscr{H}_1 = 0$, then the control may be equal to zero, and if $\mathscr{H}_2 = 0$, then the control must have a jump.

4. 1D IMPULSE CONTROL PROBLEM

In the case of one-dimensional state space ($x \in \mathbb{R}^1$) it is possible to present an explicit expression for the value function.

4.1 Problem Formulation

Here we consider the impulse control problem in $\mathbb{R}$ with a specific terminal functional:

$$dx(t) = b(t)dU(t) + c(t)v(t)dt, \quad t \in [t_0, t_1],$$
$$\operatorname{Var} U(\cdot) + \alpha_0 d(x(t_1+0), M) + k_0 \to \inf,$$

where $M = [m_1, m_2]$, $v(t) \in \mathscr{Q}(t)$, $\mathscr{Q}(t) = [q_1(t), q_2(t)]$.

4.2 Min-Max Value Function

Denote the min-max value function as

$$V^-(t_0,x_0) = \min_{U(\cdot)} \max_{v(\cdot)} [\operatorname{Var} U(\cdot) + \alpha_0 d(x(t_1+0), M) + k_0 \,|\, x(t_0) = x_0].$$

It can be rewritten as follows

$$V^-(t_0,x_0) = \min_{x_1} \left[\min_{U(\cdot)} \big[\max_{v(\cdot)} [\operatorname{Var} U(\cdot) + \alpha_0 d(x(t_1+0), M) + k_0] \,\big|\, x(t_1) = x_1 \big] \,\Big|\, x(t_0) = x_0 \right].$$

Theorem 3. Min-max value function $V^-(t,x)$ belongs to the class of functions $\alpha d(x,N) + k$ and may be expressed in the following form:

$$V^-(t_0,x_0) = \min(\alpha_{[t_0,t_1]}, \alpha_0) d(x_0, N) + \alpha_0 k + k_0,$$

where

$$\alpha_{[t_0,t_1]} = \min_{t\in[t_0,t_1]} |b(t)|^{-1},$$
$$N = [n_1, n_2], \quad n_i = m_i - q_i, \quad i = 1,2,$$
$$k = \frac{1}{2}(n_1 - n_2) - \frac{1}{2}\left(\int_{t_0}^{t_1} c(\xi) q_1(\xi) d\xi - \int_{t_0}^{t_1} c(\xi) q_2(\xi) d\xi \right).$$

Proof. We first introduce three lemmas.

Lemma 4. For the linear system $dx(t) = b(t)dU(t)$:

$$\min[\operatorname{Var} U(\cdot) \,|\, x(t_0) = x_0, x(t_1) = x_1] = \frac{|x_1 - x_0|}{\max_{t\in[t_0,t_1]} |b(t)|}.$$

Lemma 5. Let $M = [m_1, m_2]$, $\mathscr{Q} = [q_1, q_2]$, then

$$\max_{\mathbf{v}\in\mathscr{Q}} d(x + \mathbf{v}, M) = \begin{cases} d(x, [n_1, n_2]), & \text{if } q_2 - q_1 < m_2 - m_1, \\ d(x, m^*) + k^*, & \text{otherwise.} \end{cases}$$

Here

$$n_1 = m_1 - q_1, \quad n_2 = m_2 - q_2,$$
$$m^* = \frac{m_1 + m_2}{2} - \frac{q_1 + q_2}{2},$$
$$k^* = \frac{m_1 - m_2}{2} - \frac{q_1 - q_2}{2}.$$

In the second case the interval $N = [n_1, n_2]$ contains only one point $\{m^*\}$.

Lemma 6. Let $\alpha > 0$, $\beta > 0$, $N = [n_1, n_2]$. Then

$$\min_{x_1} \{\beta d(x_1, N) + k + \alpha |x_1 - x_0|\} = \min(\alpha, \beta) d(x_0, N) + k.$$

We further use the previous lemmas to prove Theorem 3. Consider the min-max value function

$$V^-(t_0,x_0) = \min_{U(\cdot)} \max_{v(\cdot)} [\operatorname{Var} U(\cdot) + \alpha_0 d(x(t_1+0), M) + k_0 \,|\, x(t_0) = x_0].$$

The right end of the trajectory $x(t_1+0)$ may be expressed in two terms:

$$x(t_1+0) = \left(x(t_0) + \int_{t_0}^{t_1} b(\xi) dU(\xi) \right) + \left(\int_{t_0}^{t_1} c(\xi) v(\xi) d\xi \right) = x_1 + \mathbf{v},$$

where $\mathbf{v}$ belongs to the set $\left[\int_{t_0}^{t_1} c(t) q_1(t) dt, \int_{t_0}^{t_1} c(t) q_2(t) dt \right]$.

According to the lemmas of the above we calculate the value function as

$$V^-(t_0,x_0)=\min_{x_1}\Bigg[\min_{U(\cdot)}\big[\max_{v(\cdot)}[\alpha_0 d(x_1+\mathbf{v},M)+k_0+$$
$$+\operatorname{Var}U(\cdot)]\;\big|x(t_1)=x_1\big]\;\Bigg|x(t_0)=x_0\Bigg]=$$
$$=\min_{x_1}\big[\min_{U(\cdot)}[\alpha_0\left(d(x_1,N)+k\right)+k_0+$$
$$+\operatorname{Var}U(\cdot)\;\big|x(t_1)=x_1\big]\;\big|x(t_0)=x_0\big]=$$
$$=\min_{x_1}[\alpha_0 d(x_1,N)+\alpha_0 k+k_0+$$
$$+\alpha_{[t_0,t_1]}|x_1-x_0|\;\big|x(t_0)=x_0\big]=$$
$$=\min(\alpha_{[t_0,t_1]},\alpha_0)d(x_0,N)+\alpha_0 k+k_0.$$

Therefore, the minimax value function $V^-(t,x)$ belongs to the class of functions $\alpha d(x,N)+k$.

4.3 Value function with corrections

We recurrently define min-max value functions with k corrections at $\tau_1,\dots,\tau_k$, where $t_0<\tau_k<\dots<\tau_1<t_1$.

The value function with zero corrections is
$$V^-=\min_{x_1}\min_{U(\cdot)}\max_{\mathbf{v}(\cdot)}\left[Var\,U(\cdot)+\alpha_0 d(x_1+\mathbf{v}(\cdot),M)+k_0\right].$$

We make one correction at τ_1: the right end of the trajectory $x(t_1)$ may be expressed as
$$x(t_1)=x_1+\mathbf{v}$$
$$x_1=x_{\tau_1}+\int_{\tau_1}^{t_1}b(\xi)dU(\xi),\quad \mathbf{v}=\int_{\tau_1}^{t_1}c(\xi)v(\xi)d\xi,$$
where $\mathbf{v}$ is it the set $\left[\int_{\tau_1}^{t_1}c(\xi)q_1(\xi)d\xi,\int_{\tau_1}^{t_1}c(\xi)q_2(\xi)d\xi\right]$.

It may be shown that the value function with one correction at τ_1 belongs to the same class of functions and is expressed as $V_0^-=\alpha_1 d(x_{\tau_1},N^1)+k^1$, where α_1, k^1 and interval N^1 are explicitly calculated from the given data.

Then we continue by introducing a correction at τ_2, and further, using V_1^- as a terminal function, to create value function V_2^- with two corrections. We continue still further similarly.

4.4 Close-Loop Value Function

As we proceed, increasing the number of corrections towards infinity, in the limit we come to the value function expressed in a similar way, namely,
$$\mathscr{V}^-=\alpha d(x_0,N)+k.$$
Here, the interval N contains more than one point
$$N=\left[m_1-\int_{t_0}^{t_1}c(\xi)q_1(\xi)d\xi,m_2-\int_{t_0}^{t_1}c(\xi)q_2(\xi)d\xi\right],$$
and $k=k_0$, if
$$m_2-m_1>\int_{t_0}^{t_1}c(\xi)q_2(\xi)d\xi-\int_{t_0}^{t_1}c(\xi)q_1(\xi)d\xi.$$

Otherwise, we have a degenerate case when N contains only one one point
$$N=n^*=\frac{1}{2}\left(m_1+m_2\right)-\frac{1}{2}\Big(\int_{t_0}^{t_1}c(\xi)q_1(\xi)d\xi+$$
$$+\int_{t_0}^{t_1}c(\xi)q_2(\xi)d\xi\Big).$$
$$k=\frac{1}{2}\,\alpha\int_{t_0}^{\tau^*}c(t)(q_2(t)-q_1(t))dt,$$
τ^* is the time moment when
$$m_2-m_1=\int_{\tau^*}^{t_1}c(\xi)q_2(\xi)d\xi-\int_{\tau^*}^{t_1}c(\xi)q_1(\xi)d\xi.$$
In both cases
$$\alpha=\min\big(\alpha_0,\min_{t\in[t_0,t_1]}|b(t)|^{-1}\big).$$

For the value function $\mathscr{V}^-=\alpha d(x_0,N)+k$ the semi-group property is satisfied.

4.5 Example

Consider a linear system[1] $dx=(1-t^2)dU+v(t)dt$ with $[t_0,t_1]=[-1,1]$, $M=0$, where the disturbance $v(t)\in[-1,1]$. It has to be steered from its initial state $x(-1)=x$ by the control that delivers a minimum to functional
$$\operatorname{Var}U(\cdot)+2d(x(t_1+0),M)\to\inf. \tag{6}$$

The value function $\mathscr{V}^-(t,x)=\alpha(t)|x|$, where
$$\alpha(t)=\min\left(2,\min_{\tau\in[t,1]}\frac{1}{1-\tau^2}\right).$$

We calculate the Hamiltonian functions:
$$\mathscr{H}_1=\begin{cases}\dfrac{tx}{1-t^2}, & \text{if } 0\le t\le 1/\sqrt{2},\\ 0, & \text{if } -1\le t<0, \text{ and } 1/\sqrt{2}<t\le 1.\end{cases}$$
$$\mathscr{H}_2=\begin{cases}t^2, & \text{if } -1\le t<0,\\ 2t^2-1, & \text{if } 1/\sqrt{2}<t\le 1,\\ 0, & \text{if } 0\le t\le 1/\sqrt{2}.\end{cases}$$

There are three cases:

(1) if $t<0$ we have $\mathscr{H}_1=0$, $\mathscr{H}_2\neq 0$, then we do not apply control;
(2) if $0\le t\le 1/\sqrt{2}$, we have $\mathscr{H}_1\neq 0$, $\mathscr{H}_2=0$ and we steer our system with an impulse control;
(3) if $1/\sqrt{2}<t\le 1$, we have $\mathscr{H}_1=0$, $\mathscr{H}_2\neq 0$, then we do not apply control.

The control $U=-\gamma\delta(t-t^*)$, where $\gamma=\dfrac{x(t^*-0)}{1-t^2}$, because we need to reach $x(t^*+0)=0$.

Figs. 1, 2 show trajectories $x(t)$ and control U for different disturbance $v(t)$. Note that we apply impulse control when $0\le t^*\le\dfrac{1}{\sqrt{2}}$ and the trajectory reaches zero. After that we do not apply control, and the trajectory drifts away from zero, because of the disturbance. This is the trajectory and the control that deliver minimum to functional (6).

5. FAST CONTROLS

Impulse control is an “ideal” one. Bounded functions approximating impulse controls are known as *fast controls*, since they are physically realizable and may steer a system

[1] The calculations in this example were performed by Anastasia Melnikova.

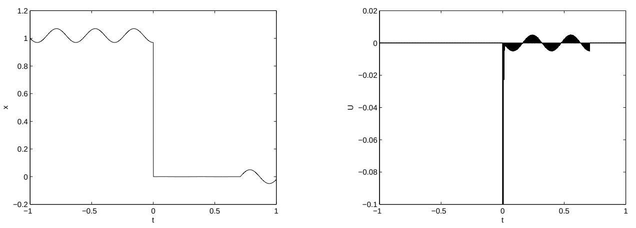

Fig. 1. Trajectory of the system, starting from $x(-1) = 1$, and corresponding control. Disturbance is $v(t) = \sin(20t)$.

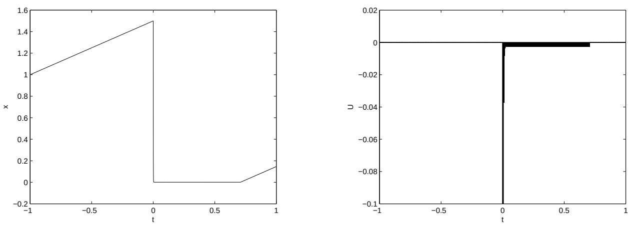

Fig. 2. Trajectory of the system, starting from $x(-1) = 1$, and corresponding control. Disturbance is constant $v(t) = 1$.

to a given state in arbitrary small time. Such controls may be found, for example, in the following form:

$$u_\Delta(t) = \sum\nolimits_{j=0}^{m} u_j \Delta_{h_j}^{(j)}(t-\tau), \tag{7}$$

where $\Delta_h^{(j)}(t)$ approximate the derivatives of delta-function:

$$\begin{gathered} \Delta_h^{(0)}(t) = h^{-1}\mathbf{1}_{[0,h]}(t), \\ \Delta_h^{(j)}(t) = h^{-1}\left(\Delta_h^{(j-1)}(t) - \Delta_h^{(j-1)}(t-h)\right). \end{gathered} \tag{8}$$

The next problem is how to choose the parameters of control (7) — the coefficients h_j and vectors u_j. These should be chosen following physical requirements on the realizations of the control.

5.1 Discontinuous Approximations

We first consider fast controls with various restrictions:

(1) bounded time of control:
$$\max_j\{(j+1)h_j\} \le H;$$
(2) hard bounds on control:
$$\|u_\Delta(t)\| \le \mu;$$
(3) separate hard bounds on approximations of generalized functions of all orders included in the control:
$$\|u_{\Delta,j}(t)\| \le \mu_j,$$
$$u_{\Delta,j}(t) = u_j \Delta_{h_j}^{(j)}(t-\tau).$$

The indicated restrictions lead to moment problems of similar type.

$$\begin{gathered} \mu \to \inf, \\ \left|\Delta_h^{(n)}(t)\right| \le \mu, \quad t \in [-h, h]. \end{gathered} \tag{9}$$

We impose extra restrictions to ensure that the approximations $\Delta_h^{(n)}(t)$ affect polynomials of degree n in the same way that $\delta^{(n)}(t)$.

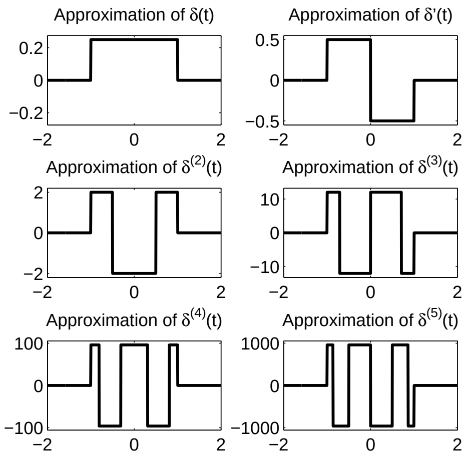

Fig. 3. Discontinuous approximations of $\delta(t)$, $\delta'(t), \ldots, \delta^{(5)}(t)$ with minimal modulus on fixed time interval.

$$\begin{aligned} &\int_{-h}^{h} \Delta_h^{(n)}(t) t^k dt = 0, \quad k = 0, \ldots, n-1, \\ &\int_{-h}^{h} \Delta_h^{(n)}(t) t^n dt = (-1)^n n! \end{aligned} \tag{10}$$

The moment problem (9) with restrictions (10) has the following solution:

$$\Delta_h^{(n)}(t) = \tfrac{1}{4}(-1)^n n! \left(\tfrac{2}{h}\right)^{(n+1)} \operatorname{sign} U_n(ht), \tag{11}$$

where $U_n(t)$ is the Chebyshev polynomial of the second kind: $U_n(t) = \cos(n \arccos t)$.

Approximation (11) is piecewise constant (and hence discontinuous), equal to $\pm\frac{1}{4}n!\left(\frac{2}{h}\right)^{(n+1)}$ between Chebyshev points $t_k = h\cos\frac{\pi j}{n+1}$, $j = 0, \ldots, n+1$. See Fig. 3.

5.2 Smooth Approximations

Apart from discontinuous, we also consider continuous or smooth approximations. To do this, we impose bounds on the k-th derivatives of the approximation:

$$\Delta_{h,k}^{(n)}(t) = \int_{-h}^{t}\int_{-h}^{t_1}\cdots\int_{-h}^{t_{k-1}} g_k^n(t_k)\,dt_k dt_{k-1}\ldots dt_1,$$

$$|g_k^n(t)| \le \mu.$$

And we add similar restrictions on related polynomials of degree n, that were used for discontinuous approximations:

$$\int_{-h}^{h} \Delta_{h,k}^{(n)}(t) t^j dt = 0, \quad j = 0, \ldots, n-1,$$
$$\int_{-h}^{h} \Delta_{h,k}^{(n)}(t) t^n dt = (-1)^n n!$$

This leads to moment problems for the k-th derivative $g_k^n(t)$ of approximation $\Delta_{h,k}^{(n)}(t)$:

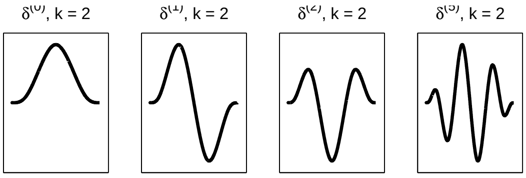

Fig. 4. Continuously differentiable approximations of $\delta(t)$ and its derivatives.

$$\mu \to \inf,$$
$$|g_k^n(t)| \le \mu, \quad t \in [-h, h],$$

$$\int_{-h}^{h} g_k^n(t) t^j dt = 0, \quad j = 0, \dots, n+k-1,$$
$$\int_{-h}^{h} g_k^n(t) t^{n+k} dt = (-1)^{n+k}(n+k)!$$

It turns out that a $(k-1)$-times smooth approximation of $\delta^{(n)}(t)$, $\Delta_{h,k}^{(n)}(t)$, is a normalized k-fold integral of $\Delta_h^{(n+k)}(t)$:

$$\Delta_{h,k}^{(n)}(t) = \frac{1}{(k-1)!} \int_{-h}^{t} g_k^n(\tau)(t-\tau)^{k-1} d\tau, \qquad (12)$$

$$g_k^n(t) = \Delta_h^{(n+k)}(t) =$$
$$= \tfrac{1}{4}(-1)^{n+k} \left(\tfrac{2}{h}\right)^{n+k+1} (n+k)! \mathrm{sign} U_{n+k}(ht).$$

Here $k = -1$ corresponds to discontinuous approximations $\Delta_h^{(n)}(t)$, and $k = 0$ leads to continuous (but not smooth) approximations.

Approximations $\Delta_{h,k}^{(n)}(t)$ are piecewise polynomials of order k, with $k-1$ derivatives continuous at the junction points. The coefficients of these polynomials may be calculated recursively by explicit formulae.

In Fig. 4 we present our continuously differentiable approximations of $\delta(t)$ and its derivatives.

5.3 Growth Rate of Fast Controls

Here we present some estimates on how fast do the norms μ of a fast controls grow with time interval h tending to zero. We assume that the aim of the control is to steer the system to the origin.

Suppose that $A(t) \equiv A$, $B(t) \equiv B$. According to Seidman and Yong (1997), the minimum variation of the impulse control is varying asymptotically as

$$\mu \sim h^{-r}, \quad r = \min\{j \mid x_0 \in R_j\}, \qquad (13)$$

where $R_j = \mathrm{im}\left(B \ AB \ \cdots A^j B\right)$.

It was shown by Daryin and Kurzhanski (2008) that for fast controls of type (8) the estimate (13) holds, with $R_j = \left(F_h^{(0)} B \ F_h^{(1)} B \ \cdots \ F_h^{(j)} B\right)$, $F_h^{(s)} = h^{-s}(1-e^{-hA})^s F_h$, and $F_h = h^{-1}\int_0^h e^{-tA} dt$.

Using the same reasoning, one comes to the estimate (13) for fast controls of type (11), with

$$R_j = \left(\hat{F}_h^{(0)} B \ \hat{F}_h^{(1)} B \ \cdots \ \hat{F}_h^{(j)} B\right),$$
$$\hat{F}_h^{(k)} = \int_{-h}^{h} \Delta_h^{(k)}(t) e^{(h-t)A} \, dt.$$

Similarly, for smooth approximations of $(k-1)$-th order of type (12), we have the following estimate:

$$\mu \sim h^{-(r+k)}, \quad r = \min\{j \mid x_0 \in R_j\}. \qquad (14)$$

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