Closed-Loop Impulse Control of Oscillating Systems

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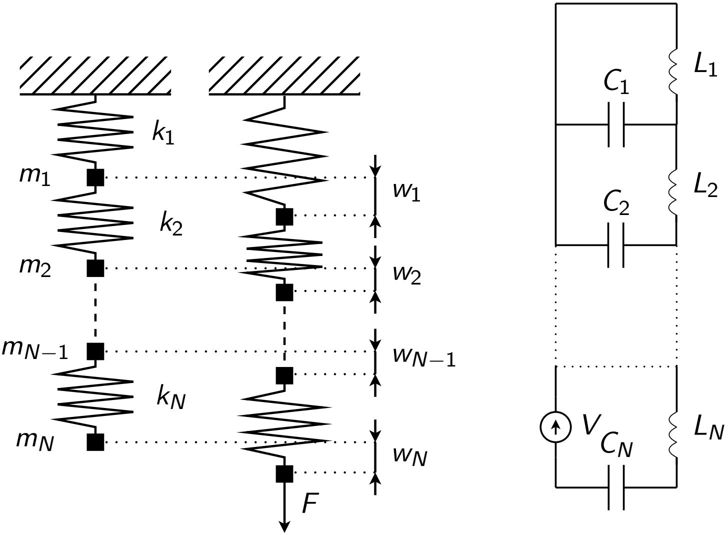
Outline

Oscillating System



Oscillating System

$$\begin{cases} m_1 \ddot{w}_1 = k_2(w_2 - w_1) - k_1 w_1 \\ m_i \ddot{w}_i = k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}) \\ m_\nu \ddot{w}_\nu = k_{\nu+1}(w_{\nu+1} - w_\nu) - k_\nu(w_\nu - w_{\nu-1}) + u(t) \\ m_N \ddot{w}_N = -k_N(w_N - w_{N-1}) \end{cases}$$

- $w_i = w_i(t)$ — displacements from the equilibrium
- m_i — masses of the loads
- k_i — stiffness coefficients
- $u(t) = \frac{dU}{dt}$ — impulse control $(U \in BV)$

$N \to \infty$

$$\rho(\xi) w_{tt}(t,\xi) = [Y(\xi) w_\xi(t,\xi)]_\xi, \quad t > t_0, \quad 0 < \xi < L$$

$$w(t,0) = 0, \quad w_\xi(t,L) = u(t)/Y(L), \quad t \geqslant t_0$$

$$w(t_0,\xi) = w^0(\xi), \quad w_t(t_0,\xi) = \dot{w}^0(\xi), \quad 0 \leqslant \xi \leqslant L$$

- $w(t,\xi)$ — displacement from the equilibrium
- $u(t) = \frac{dU}{dt}$ — impulse control
- $\rho(\xi)$ — mass density
- $Y(\xi)$ — Young modulus

Oscillating System

Normalized matrix form:

$$dx(t) = Ax(t)dt + B\,dU(t)$$

$$x(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \quad w(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_N(t) \end{pmatrix}$$

This system is **completely controllable**.

Impulse Control Problem

Problem (1)

$$\textit{Minimize } J(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1+0))$$

over $U(\cdot) \in BV[t_0, t_1]$ *where* $x(t)$ *is the trajectory generated by control input*

$$u(t) = \frac{\mathrm{d}U}{\mathrm{d}t}$$

starting from $x(t_0 - 0) = x_0$.

$$u(t) = \sum_{i=1}^{2N} h_i \delta(t - \tau_i)$$

Important particular case: $\varphi(x) = \mathscr{I}(x \mid \{0\})$

— completely stop oscillations on fixed time interval $[t_0, t_1]$.

The Value Function

Definition

The minimum of $J(U(\cdot))$ with *fixed* initial position $x(t_0 - 0) = x_0$ is called the value function:

$$V(t_0, x_0) = V(t_0, x_0; t_1, \varphi(\cdot)).$$

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathbb{R}^n} \frac{\left\langle p, x_1 - \mathrm{e}^{(t_1 - t_0)A} x_0 \right\rangle}{\left\| B^T \mathrm{e}^{(t_1 - \cdot)A^T} p \right\|_{C[t_0, t_1]}} \right\}.$$

The value function is convex and its conjugate equals

$$V^*(t_0, p) = \varphi^*(\mathrm{e}^{(t_0 - t_1)A^T} p) + \mathscr{I}\left(\mathrm{e}^{(t_0 - t_1)A^T} p \;\middle|\; \mathscr{B}_{\|\cdot\|_{[t_0, t_1]}} \right)$$

where $\|p\|_{[t_0, t_1]} = \left\| B^T \mathrm{e}^{(t_1 - \cdot)A^T} p \right\|_{C[t_0, t_1]}$.

Dynamic Programming Equation

The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies
the Principle of Optimality

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))), \quad \tau \in [t_0, t_1]$$

The value function it is the solution to the
Hamilton–Jacobi–Bellman quasi-variational inequality:

$$\min \{H_1(t, x, V_t, V_x),\ H_2(t, x, V_t, V_x)\} = 0,$$

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)).$$

$$H_1 = V_t + \langle V_x, Ax \rangle, \quad H_2 = \min_{u \in S_1} \langle V_x, Bu \rangle + 1 = -\left\| B^T V_x \right\| + 1.$$

The Control Structure

$$H_1(t,x)=0 \longleftarrow (t,x) \longrightarrow H_2(t,x)=0$$

Branch $H_1(t,x)=0$:

↓

wait

$$dU(t)=0$$

Branch $H_2(t,x)=0$:

↓

choose jump direction

$$d=-B^T V_x$$

↓

choose jump amplitude

$$\min \alpha \geqslant 0 : H_1(t, x+\alpha d)=0$$

↓

jump

$$U(\tau)=\alpha \cdot d \cdot \chi(\tau - t)$$

Numerical Algorithm

The value function is

$$V(t_0, x_0) = \max_{\substack{p \in \mathbb{R}^n \\ \left\| B^T \mathrm{e}^{(t_1 - t)A^T} p \right\| \leqslant 1 \\ \forall t \in [t_0, t_1]}} \langle p, x_0 \rangle .$$

Replace $\left\| B^T \mathrm{e}^{(t_1 - t)A^T} p \right\| \leqslant 1$ by a finite number of linear inequalities, and $[t_0, t_1]$ with a finite number of time instants:

$$\hat{V}(t_0, x_0) = \max_{\substack{p \in \mathbb{R}^n \\ \left\langle q_i, B^T \mathrm{e}^{(t_1 - t)A^T} p \right\rangle \leqslant 1, i = \overline{1, M} \\ t = \theta_1, \theta_2, \ldots, \theta_K}} \langle p, x_0 \rangle$$

which is a LP problem.

Numerical Algorithm

Finding control for given (t, x) is a LP ranging problem.

The error estimate is

$$V(t,x) \leqslant \hat{V}(t,x) \leqslant V(t,x)\big[1 + O\big(K^{-1}\big)\big],$$

Ellipsoidal Approximation

$\mathscr{X}_\nu[t]$ — backward reach set under condition $\operatorname{Var} U \leqslant \nu$

$$V(t,x) = \min\{\nu \mid x \in \mathscr{X}_\nu[t]\}$$

We look for an approximation of $\mathscr{X}_\nu[t]$.

Ellipsoids:

$$\mathscr{E}(q,Q) = \left\{ x \mid \left\langle x-q, Q^{-1}(x-q)\right\rangle \leqslant 1 \right\}$$
$$\rho(\ell \mid \mathscr{E}(q,Q)) = \langle \ell, q\rangle + \langle \ell, Q\ell\rangle^{\frac{1}{2}}$$

(see Kurzhanski and Vályi, 1997)

Ellipsoidal Approximation

Ellipsoidal approximation is derived through comparison principle for Hamilton–Jacobi equations (Kurzhanski, 2006):

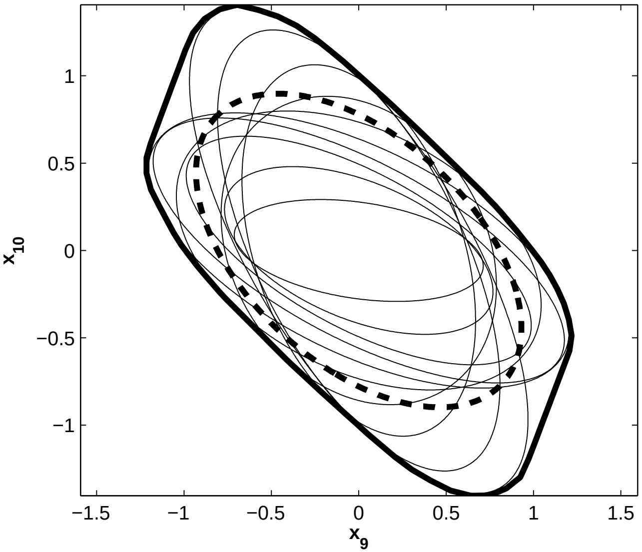
$$\mathscr{X}_{\nu}^{-}[t] = \mathscr{E}(0, (\nu - k(t))Z(t))$$

$$\begin{cases} \dot{Z} &= AZ + ZA^{T} - \eta(t)BB^{T} \\ \dot{k} &= -\frac{1}{4}\eta(t) \end{cases} \qquad \begin{cases} Z(t_1) &= 0 \\ k(t_1) &= 0 \end{cases}$$

Here $\eta(t) \geqslant 0$ is a parameter function

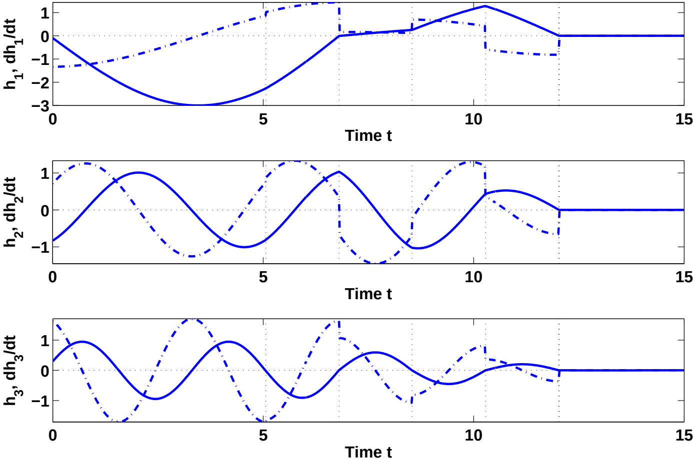
$$\mathscr{X}_{\nu}[t] = \mathrm{cl} \bigcup_{\nu(\cdot)} \mathscr{X}_{\nu}^{-}[t]$$

Ellipsoidal Approximation



Asymptotic Solution ($\Delta t \to \infty$)

$$\ddot{h}_i = -\omega_i^2 h_i + b_i u, \quad i = \overline{1, N}.$$



Asymptotic Solution ($\Delta t \to \infty$)

$$\operatorname{cl} \bigcup_{t \leqslant t_1} \mathscr{X}_1[t] = \mathscr{C} = \bigcap_{j=1}^{N} C_j$$

- $\mathscr{X}_1[t]$ — backward reach set under condition $\operatorname{Var} U \leqslant 1$
- $C_j = \left\{ (h, \dot{h}) \,\middle|\, \omega_j^2 h_j^2 + \dot{h}_j^2 \leqslant b_j^2 \right\}$ — ellipsoids

$$V \underset{t \to -\infty}{\to} \mathscr{V} = \max_{j=\overline{1,N}} \sqrt{\frac{\omega_j^2 h_j^2 + \dot{h}_j^2}{b_j^2}} \qquad (*)$$

Control strategy:

- "Optimal": jump if $\left| B^T \mathscr{V}_x \right| = 1 \iff h_j = 0$ for all maximizers j in (*). Useless after first jump.
- ε-optimal: jump if $\left| B^T \mathscr{V}_x \right| \geqslant 1 - \varepsilon$:

$$\operatorname{Var} U(\cdot) \leqslant \frac{\mathscr{V}}{1 - \varepsilon}$$

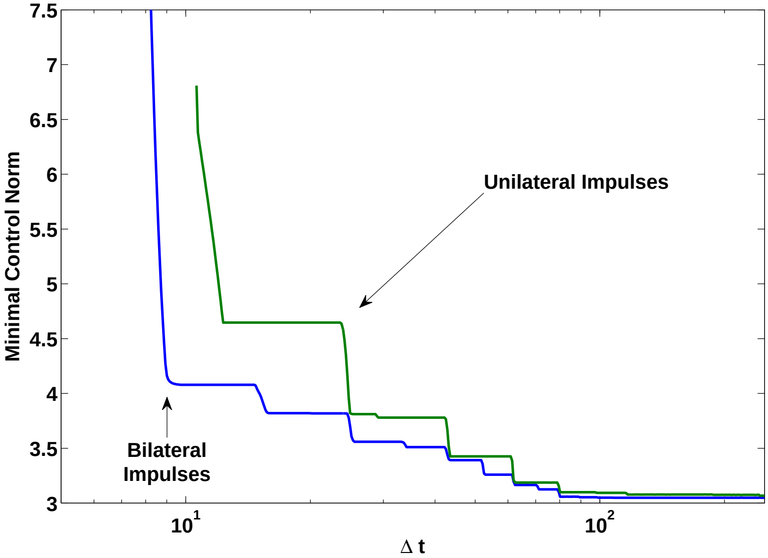
Unilateral Impulses

Additional constraint: $dU \geqslant 0$ ($dU \leqslant 0$).

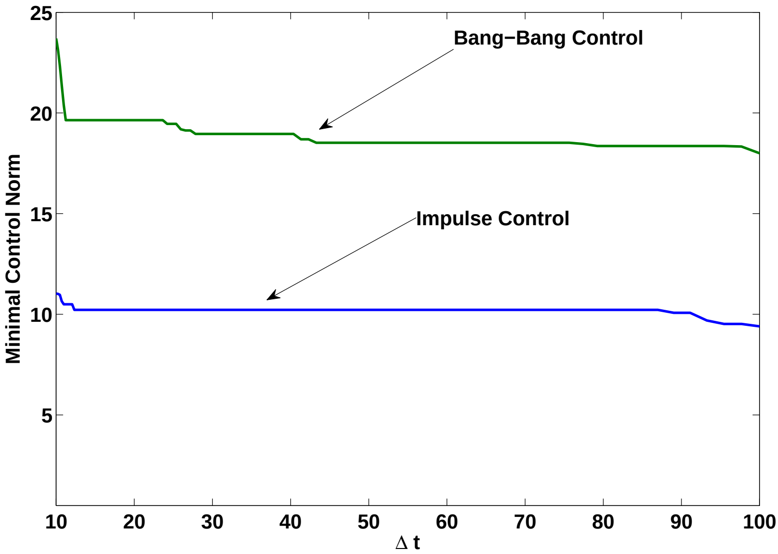
General case: $KdU \geqslant 0$ (K — matrix) or $dU \in \mathscr{K}$ ($\mathscr{K}$ — cone).

- The minimum number of impulses is the same — $2N$.
- Numerical procedures apply with minor modifications.
- Asymptotic solution does not change.
- The problem may be not solvable on small time intervals.

Unilateral Impulses



Impulse vs Bang-Bang Controls



Double Constraint Approach

Problem (2)

$$\textit{Minimize } J(u) = \int_{t_0}^{t_1} |u(t)|\, dt + \varphi(x(t_1))$$

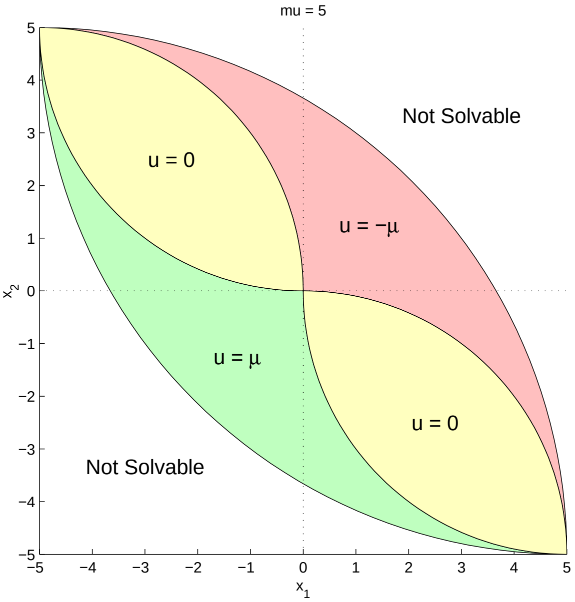
over controls $u(t)$ satisfying $|u(t)| \leqslant \mu$, where $x(t)$ is the trajectory generated by control u starting from $x(t_0) = x_0$.

Here controls are bounded functions.
Optimal controls only take values $-\mu$, 0, μ.

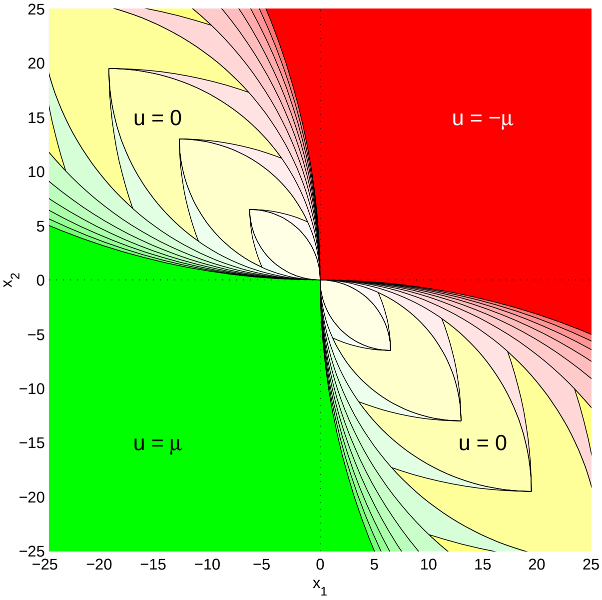
$V_\mu(t, x)$ is the value function for Problem 2.

$$0 \leqslant V_\mu(t, x) - V(t, x) = O(\mu^{-1}) \qquad \text{for each } (t, x)$$

Double Constraint Approach



Double Constraint Approach



Generalized Impulse Control Problem

Problem (3)

$$\textit{Minimize } J(u) = \rho^*[u] + \varphi(x(t_1 + 0))$$

over distributions $u \in D_1[\alpha, \beta]$, $(\alpha, \beta) \supseteq [t_0, t_1]$ *where* $x(t)$ *is the trajectory generated by control* u *starting from* $x(t_0 - 0) = x_0$.

Here $\rho^*[u]$ is the conjugate norm to the norm ρ on $C^1[\alpha, \beta]$:

$$\rho[\psi] = \max_{t \in [\alpha, \beta]} \sqrt{\|\psi(t)\|^2 + \|\psi'(t)\|^2}.$$

$$u(t) = \sum_{i=1}^{2N} h_i^{(0)} \delta(t - \tau_i) + h_i^{(1)} \delta'(t - \tau_i).$$

Reduction to Impulse Control Problem

$$u \in D_1 : \quad \boxed{u = \frac{\mathrm{d}U_0}{\mathrm{d}t} + \frac{\mathrm{d}^2 U_1}{\mathrm{d}t^2}} \quad U_0, U_1 \in BV$$

Problem 3 reduces to a particular case of Problem 1 for the system

$$\dot{x} = Ax + \mathscr{B}u, \quad \mathscr{B} = \begin{pmatrix} B & AB \end{pmatrix}$$

and the control

$$u = \frac{\mathrm{d}U}{\mathrm{d}t}, \quad U(t) = \begin{pmatrix} U_0(t) \\ U_1(t) \end{pmatrix}.$$

Error bound for numerical algorithm:

$$V(t,x) \leqslant \hat{V}(t,x) \leqslant V(t,x)\left[1 + O\left(K^{-1} + M^{-2}\right)\right],$$

Examples

- Chain of 5 springs
- String (10 elements)

References

- Bellman, R. (1957). *Dynamic Programming*. Princeton Univ. Press.
- Bensoussan, A. and J.-L. Lions (1982). *Contrôle impulsionnel et inéquations quasi variationnelles*. Paris.
- Crandall, M. G. and P.-L. Lions (1983). Viscosity solutions of Hamilton–Jacobi equations. *Trans. Amer. Math. Soc.* **277**, 1–41.
- Daryin, A. N., A. B. Kurzhanski and A. V. Seleznev (2005). A dynamic programming approach to the impulse control synthesis problem. In: *Proc. Joint 44th IEEE CDC-ECC 2005*. IEEE. Seville.
- Demyanov, V. F. (1974). *Minimax: Directional Derivates*. Nauka. Moscow.
- Dykhta, V. A. and O. N. Sumsonuk (2003). *Optimal impulsive control with applications*. Fizmatlit. Moscow.
- Gusev, M. I. (1975). On optimal control of generalized processes under non-convex state constraints. In: *Differential Games and Control Problems*. UNC AN SSSR. Sverdlovsk.
- Kalman, R. E. (1960). On the general theory of control systems. In: *Proc. 1st IFAC Congress*. Vol. 1. IFAC. Butterworths. London.
- Krasovski, N. N. (1957). On a problem of optimal regulation. *Prikl. Math. & Mech.* **21**(5), 670–677.

References

- Krasovski, N. N. (1968). *The Theory of Motion Control*. Nauka. Moscow.
- Kurzhanski, A. B. (1975). Optimal systems with impulse controls. In: *Differential Games and Control Problems*. UNC AN SSSR. Sverdlovsk.
- Kurzhanski, A. B. and I. Vályi (1997). *Ellipsoidal Calculus for Estimation and Control*. SCFA. Birkhäuser. Boston.
- Kurzhanski, A. B. and Yu. S. Osipov (1969). On controlling linear systems through generalized controls. *Differenc. Uravn.* **5**(8), 1360–1370.
- Miller, B. M. and E. Ya. Rubinovich (2003). *Impulsive Control in Continuous and Discrete-Continuous Systems*. Kluwer. N.Y.
- Motta, M. and F. Rampazzo (1995). Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential and Integral Equations* **8**, 269–288.
- Neustadt, L. W. (1964). Optimization, a moment problem and nonlinear programming. *SIAM J. Control* **2**(1), 33–53.
- Rockafellar, R. T. (1970). *Convex Analysis*. Vol. 28 of *Princeton Mathematics Series*. Princeton University Press.

Continuous and Smooth Controls

It is required to use continuous or smooth controls.

Control force is produced by an integrator

$$F(t) = \underbrace{\int_{t_0}^{t} \int_{t_0}^{\tau_\nu} \cdots \int_{t_0}^{\tau_2}}_{\nu \text{ times}} u(\tau_1)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\nu$$

$u(t)$ is the new control variable.

Hard bound (geometrical constraint) on control:

$$u(t) \in \mathscr{P} = [-\mu, \mu]$$

Examples:

- Continuous Control
- Smooth Control