Closed-Loop Impulse Control of Oscillating Systems

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Oscillating System
Oscillating System

\[
\begin{align*}
    m_1 \ddot{w}_1 &= k_2 (w_2 - w_1) - k_1 w_1 \\
    m_i \ddot{w}_i &= k_{i+1} (w_{i+1} - w_i) - k_i (w_i - w_{i-1}) \\
    m_\nu \ddot{w}_\nu &= k_{\nu+1} (w_{\nu+1} - w_\nu) - k_\nu (w_\nu - w_{\nu-1}) + u(t) \\
    m_N \ddot{w}_N &= -k_N (w_N - w_{N-1})
\end{align*}
\]

- \( w_i = w_i(t) \) — displacements from the equilibrium
- \( m_i \) — masses of the loads
- \( k_i \) — stiffness coefficients
- \( u(t) = \frac{dU}{dt} \) — impulse control \((U \in BV)\)
\[ N \to \infty \]

\[ \rho(\xi)w_{tt}(t, \xi) = [Y(\xi)w_\xi(t, \xi)]_\xi, \quad t > t_0, \quad 0 < \xi < L \]

\[ w(t, 0) = 0, \quad w_\xi(t, L) = u(t)/Y(L), \quad t \geq t_0 \]

\[ w(t_0, \xi) = w^0(\xi), \quad w_t(t_0, \xi) = \dot{w}^0(\xi), \quad 0 \leq \xi \leq L \]

- \( w(t, \xi) \) — displacement from the equilibrium
- \( u(t) = \frac{dU}{dt} \) — impulse control
- \( \rho(\xi) \) — mass density
- \( Y(\xi) \) — Young modulus
Normalized matrix form:

\[ dx(t) = Ax(t)dt + B dU(t) \]

\[ x(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \end{pmatrix} \quad w(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_N(t) \end{pmatrix} \]

This system is **completely controllable**.
Problem (1)

Minimize $J(U(\cdot)) = \text{Var} U(\cdot) + \varphi(x(t_1 + 0))$

over $U(\cdot) \in \mathbf{BV}[t_0, t_1]$ where $x(t)$ is the trajectory generated by control input

$$u(t) = \frac{dU}{dt}$$

starting from $x(t_0 - 0) = x_0$.

$$u(t) = \sum_{i=1}^{2N} h_i \delta(t - \tau_i)$$

Important particular case: $\varphi(x) = \mathcal{I}(x | \{0\})$ — completely stop oscillations on fixed time interval $[t_0, t_1]$. 
The Value Function

**Definition**

The minimum of $J(U(\cdot))$ with *fixed* initial position $x(t_0 - 0) = x_0$ is called the value function:

$$V(t_0, x_0) = V(t_0, x_0; t_1, \varphi(\cdot)).$$

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - e^{(t_1-t_0)A}x_0 \rangle}{\|B^T e^{(t_1-\cdot)A} p\|_{C[t_0,t_1]}} \right\}.$$

The value function is convex and its conjugate equals

$$V^*(t_0, p) = \varphi^* (e^{(t_0-t_1)A^T} p) + \mathcal{I} \left( e^{(t_0-t_1)A^T} p \middle| B \|\cdot\|_{[t_0,t_1]} \right)$$

where $\|p\|_{[t_0,t_1]} = \left\| B^T e^{(t_1-\cdot)A} p \right\|_{C[t_0,t_1]}.$
The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the Principle of Optimality

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))), \quad \tau \in [t_0, t_1]$$

The value function it is the solution to the Hamilton–Jacobi–Bellman quasi-variational inequality:

$$\min \left\{ H_1(t, x, V_t, V_x), H_2(t, x, V_t, V_x) \right\} = 0,$$

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)).$$

$$H_1 = V_t + \langle V_x, Ax \rangle, \quad H_2 = \min_{u \in S_1} \langle V_x, Bu \rangle + 1 = -\left\| B^T V_x \right\| + 1.$$
The Control Structure

\[ H_1(t, x) = 0 \quad \leftarrow \quad (t, x) \quad \rightarrow \quad H_2(t, x) = 0 \]

- wait
  - \( dU(t) = 0 \n\)

choose jump direction
  - \( d = -B^T V_x \n\)

choose jump amplitude
  - \( \min \alpha \geq 0 : H_1(t, x + \alpha d) = 0 \n\)

jump
  - \( U(\tau) = \alpha \cdot d \cdot \chi(\tau - t) \n\)
The value function is

\[ V(t_0, x_0) = \max_{p \in \mathbb{R}^n} \langle p, x_0 \rangle. \]

\[ \left\| B^T e^{(t_1 - t)A^T} p \right\| \leq 1 \]
\[ \forall t \in [t_0, t_1] \]

Replace \( \left\| B^T e^{(t_1 - t)A^T} p \right\| \leq 1 \) by a finite number of linear inequalities, and \([t_0, t_1]\) with a finite number of time instants:

\[ \hat{V}(t_0, x_0) = \max_{p \in \mathbb{R}^n} \langle p, x_0 \rangle \]
\[ \langle q_i, B^T e^{(t_1 - t)A^T} p \rangle \leq 1, i = 1, \ldots, M \]
\[ t = \theta_1, \theta_2, \ldots, \theta_K \]

which is a LP problem.
Finding control for given \((t, x)\) is a LP ranging problem.

The error estimate is

\[
V(t, x) \leq \hat{V}(t, x) \leq V(t, x)[1 + O(K^{-1})],
\]
Ellipsoidal Approximation

\( \mathcal{X}_\nu[t] \) — backward reach set under condition \( \text{Var } U \leq \nu \)

\[ V(t, x) = \min \{ \nu \mid x \in \mathcal{X}_\nu[t] \} \]

We look for an approximation of \( \mathcal{X}_\nu[t] \).

Ellipsoids:

\[ \mathcal{E}(q, Q) = \{ x \mid \langle x - q, Q^{-1}(x - q) \rangle \leq 1 \} \]

\[ \rho(\ell \mid \mathcal{E}(q, Q)) = \langle \ell, q \rangle + \langle \ell, Q\ell \rangle^{\frac{1}{2}} \]

(see Kurzhanski and Vályi, 1997)
Ellipsoidal approximation is derived through comparison principle for Hamilton–Jacobi equations (Kurzhanski, 2006):

$$\mathcal{X}_\nu^-[t] = \mathcal{E}(0, (\nu - k(t))Z(t))$$

$$\begin{cases}
\dot{Z} &= AZ + ZA^T - \eta(t)BB^T \\
\dot{k} &= -\frac{1}{4}\eta(t)
\end{cases} \begin{cases}
Z(t_1) &= 0 \\
k(t_1) &= 0
\end{cases}$$

Here $\eta(t) \geq 0$ is a parameter function

$$\mathcal{X}_\nu[t] = \text{cl} \bigcup_{\nu(\cdot)} \mathcal{X}_\nu^{-}[t]$$
Ellipsoidal Approximation
Asymptotic Solution ($\Delta t \to \infty$)

$$
\ddot{h}_i = -\omega_i^2 h_i + b_i u, \quad i = 1, N.
$$
Asymptotic Solution ($\Delta t \to \infty$)

\[
\text{cl} \bigcup_{t \leq t_1} X_1[t] = \mathcal{C} = \bigcap_{j=1}^{N} C_j
\]

- $X_1[t]$ — backward reach set under condition $\text{Var} \ U \leq 1$
- $C_j = \left\{ (h, \dot{h}) \mid \omega_j^2 h_j^2 + \dot{h}_j^2 \leq b_j^2 \right\}$ — ellipsoids

\[
V \to -\infty \quad \forall = \max_{j=1,N} \sqrt{\frac{\omega_j^2 h_j^2 + \dot{h}_j^2}{b_j^2}} \quad (*)
\]

Control strategy:
- "Optimal": jump if $|B^T \forall_x| = 1 \iff h_j = 0$ for all maximizers $j$ in (*). Useless after first jump.
- $\varepsilon$-optimal: jump if $|B^T \forall_x| \geq 1 - \varepsilon$:

\[
\text{Var} \ U(\cdot) \leq \frac{\forall}{1 - \varepsilon}
\]
Unilateral Impulses

Additional constraint: \( dU \geq 0 \) (\( dU \leq 0 \)).

General case: \( KdU \geq 0 \) (\( K \) — matrix) or \( dU \in \mathcal{K} \) (\( \mathcal{K} \) — cone).

- The minimum number of impulses is the same — \( 2N \).
- Numerical procedures apply with minor modifications.
- Asymptotic solution does not change.
- The problem may be not solvable on small time intervals.
Unilateral Impulses

Minimal Control Norm vs. $\Delta t$ for Bilateral and Unilateral Impulses.
Impulse vs Bang-Bang Controls

- **Bang-Bang Control**
- **Impulse Control**

The graph shows the minimal control norm as a function of $\Delta t$. The green line represents the Bang-Bang Control, while the blue line represents the Impulse Control. The graph indicates that Bang-Bang Control maintains a lower minimal control norm compared to Impulse Control.
Problem (2)

\begin{equation*}
\text{Minimize } J(u) = \int_{t_0}^{t_1} |u(t)| \, dt + \varphi(x(t_1))
\end{equation*}

over controls \( u(t) \) satisfying \(|u(t)| \leq \mu \), where \( x(t) \) is the trajectory generated by control \( u \) starting from \( x(t_0) = x_0 \).

Here controls are bounded functions.

Optimal controls only take values \(-\mu, 0, \mu\).

\( V_\mu(t, x) \) is the value function for Problem 2.

\[
0 \leq V_\mu(t, x) - V(t, x) = O(\mu^{-1}) \quad \text{for each } (t, x)
\]
Double Constraint Approach

\[
\begin{align*}
-5 & \quad -4 & \quad -3 & \quad -2 & \quad -1 & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 \\
\mu = 5 \\
\not\text{Solvable}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u} &= 0 \\
\mathbf{u} &= -\mu \\
\mathbf{u} &= \mu \\
\not\text{Solvable}
\end{align*}
\]
Double Constraint Approach

\[ u = \mu \]

\[ u = 0 \]

\[ u = -\mu \]
Generalized Impulse Control Problem

Problem (3)

Minimize $J(u) = \rho^*[u] + \varphi(x(t_1 + 0))$

over distributions $u \in D_1[\alpha, \beta]$, $(\alpha, \beta) \supseteq [t_0, t_1]$ where $x(t)$ is the trajectory generated by control $u$ starting from $x(t_0 - 0) = x_0$.

Here $\rho^*[u]$ is the conjugate norm to the norm $\rho$ on $C^1[\alpha, \beta]$:

$$\rho[\psi] = \max_{t \in [\alpha, \beta]} \sqrt{\|\psi(t)\|^2 + \|\psi'(t)\|^2}.$$
Reduction to Impulse Control Problem

\[ u \in D_1 : \begin{bmatrix} u = \frac{dU_0}{dt} + \frac{d^2U_1}{dt^2} \end{bmatrix} \quad U_0, U_1 \in BV \]

Problem 3 reduces to a particular case of Problem 1 for the system

\[ \dot{x} = Ax + Bu, \quad B = (B \quad AB) \]

and the control

\[ u = \frac{dU}{dt}, \quad U(t) = \begin{pmatrix} U_0(t) \\ U_1(t) \end{pmatrix}. \]

Error bound for numerical algorithm:

\[ V(t, x) \leq \hat{V}(t, x) \leq V(t, x) [1 + O(K^{-1} + M^{-2})], \]
Examples

- Chain of 5 springs
- String (10 elements)
References

References

Continuous and Smooth Controls

It is required to use continuous or smooth controls.

Control force is produced by an integrator

\[ F(t) = \int_{t_0}^{t} \int_{t_0}^{\tau_\nu} \cdots \int_{t_0}^{\tau_2} u(\tau_1) \, d\tau_1 \cdots d\tau_\nu \]

\( \nu \) times

\( u(t) \) is the new control variable.

Hard bound (geometrical constraint) on control:

\[ u(t) \in \mathcal{P} = [-\mu, \mu] \]

Examples:

- Continuous Control
- Smooth Control