Nonlinear Feedback Types in Impulse and Fast Control

Alexander N. Daryin and Alexander B. Kurzhanski

Moscow State (Lomonosov) University

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Overview

- Impulse Control System under Uncertainty
- Dynamic Programming
- Feedback Types
- Example

Impulse Control System under Uncertainty

Dynamics

$$dx(t) = A(t)x(t)dt + B(t)dU(t) + C(t)v(t)dt$$

Here

- $t \in [t_0, t_1]$ – fixed interval
- State $x(t) \in \mathbb{R}^n$
- Control $U(\cdot) \in BV([t_0, t_1]; \mathbb{R}^m)$
- Disturbance $v(t) \in \mathscr{Q}(t) \in \operatorname{conv} \mathbb{R}^k$
  - or external control

Problem

Mayer–Bolza functional:

$$J(U(\cdot), v(\cdot)) = \mathrm{Var}_{[t_0,t_1]}\, U(\cdot) + \varphi(x(t_1+0)) \to \inf$$

Problem (Impulse Control under Uncertainty)

Find a feedback control $\mathscr{U}$ minimizing the functional

$$\mathscr{J}(\mathscr{U}) = \max_{v(\cdot)\in\mathscr{Q}(\cdot)} J(U(\cdot), v(\cdot)),$$

where maximum is taken over all admissible of $v(\cdot)$ and $U(\cdot)$ is the realized impulse control.

Nonlinear Structure

The original system is linear. . .

. . . but . . .

. . . the feedback is nonlinear

$\Longrightarrow$

closed-loop system is **non-linear**
from the perspective of the external control $v(\cdot)$

Admissible open-loop controls:

$$\mathscr{C}(t) = \{U(\cdot) \in BV[t, t_1 + 0); \mathbb{R}^m \mid U(t) = 0\}.$$

Admissible disturbances:

$$\mathscr{D}(t) = \{v(\cdot) \in L_\infty[t, t_1] \mid v(s) \in \mathscr{Q}(s),\ s \in [t, t_1]\}.$$

Definition (Impulse Feedback – Non-Anticipative)

Class of **impulse feedback control strategies** $\mathscr{F}(t)$ consists of mappings $\mathscr{U} : \mathscr{D}(t) \to \mathscr{C}(t)$ such that for any $\tau \in [t, t_1]$:

$$v_1(s) \overset{\text{a.e.}}{=} v_2(s),\ s \in [t, \tau] \ \Rightarrow\ \mathscr{U}[v_1](s) \equiv \mathscr{U}[v_2](s),\ s \in [t, \tau + 0).$$

Dynamic Programming

Value Function

Definition (Value Function)

The **value function** in class of control strategies $\mathscr{F}(t)$ is

$$\begin{aligned}\mathscr{V}_{\mathscr{F}}(t,x) &= \mathscr{V}_{\mathscr{F}}(t,x;\ t_1,\varphi(\cdot)) \\ &= \inf_{\mathscr{U}\in\mathscr{F}(t)} \sup_{v\in\mathscr{D}(t)} J(\mathscr{U}[v](\cdot), v(\cdot) \mid t,x) \\ &= \inf_{\mathscr{U}\in\mathscr{F}(t)} \sup_{v\in\mathscr{D}(t)} \left\{ \mathrm{Var}_{[t,t_1+0)}\, \mathscr{U}[v](\cdot) + \varphi(x(t_1+0)) \right\}.\end{aligned}$$

$x(s)$ is the trajectory under control $\mathscr{U}[v](\cdot)$ and disturbance $v(\cdot)$.

Theorem (Principle of Optimality)

For any $\tau \in [t, t_1]$

$$\begin{aligned}\mathscr{V}_{\mathscr{F}}(t,x) &= \mathscr{V}_{\mathscr{F}}(t,x;\ \tau, \mathscr{V}_{\mathscr{F}}(\tau,\cdot)) \\ &= \inf_{\mathscr{U}\in\mathscr{F}(t)} \sup_{v\in\mathscr{D}(t)} \left\{ \mathrm{Var}_{[t,\tau+0)}\, \mathscr{U}[v](\cdot) + \mathscr{V}_{\mathscr{F}}(\tau, x(\tau+0)) \right\}.\end{aligned}$$

$\Longrightarrow$ (t,x) is the *state* of the system

Theorem (Dynamic Programming Equation)

Value function is the unique viscosity solution to

$$\min\{H_1, H_2\} = 0$$
$$\mathscr{V}(t_1, x) = V(t_1, x;\ t_1, \varphi(\cdot))$$

with Hamiltonians

$$H_1 = \max_{v \in \mathscr{Q}(t)} \mathscr{V}'(t, x \mid 1, A(t)x + C(t)v)$$
$$H_2 = \min_{\|h\|=1} \left\{ \mathscr{V}'(t, x \mid 0, B(t)h) + \|h\| \right\}$$

Theorem (Dynamic Programming Equation)

Value function is the unique viscosity solution to

$$\min\{H_1, H_2\} = 0$$
$$\mathscr{V}(t_1, x) = V(t_1, x;\ t_1, \varphi(\cdot))$$

at points of differentiability of V:

$$H_1 = V_t + \langle \mathscr{V}_x, A(t)x\rangle + \max_{v \in \mathscr{Q}(t)} \langle \mathscr{V}_x, C(t)v\rangle$$
$$= V_t + \langle \mathscr{V}_x, A(t)x\rangle + \rho\left(C^T(t)\mathscr{V}_x \,\middle|\, \mathscr{Q}(t)\right),$$

$$H_2 = \min_{\|h\|=1} \{\langle \mathscr{V}_x, B(t)h\rangle + \|h\|\} = 1 - \left\| B^T(t)\mathscr{V}_x \right\|.$$

Feedback Types

?

What is state trajectory under closed-loop control?

Here we consider the following feedback types:

0. Non-Anticipative Mapping (already discussed)
1. Formal Definition
2. Limits of Fixed-Time Impulses
3. Space-Time Transformation
4. Hybrid System
5. Constructive Motions

Definition (Impulse Feedback – Formal)

Impulse feedback control is a set-valued function $\mathscr{U}(t,x)$: $[t_0, t_1] \to \operatorname{conv}\mathbb{R}^m$, u.s.c. in (t,x), with non-empty values.

An open-loop control

$$U(t) = \sum_{j=1}^{K} h_j \chi(t - t_j)$$

conforms with $\mathscr{U}(t,x)$ under disturbance $v(t)$ if

1. for $t \neq t_j$ the set $\mathscr{U}(t, x(t))$ contains the origin;
2. $h_j \in \mathscr{U}(t_j, x(t_j))$, $j = \overline{1, K}$.
3. $\mathscr{U}(t_1, x(t_1 + 0)) = \{0\}$.

Definition (Relaxed State)

A state (t, x) is called **relaxed** if one of the following is true:

- either $t < t_1$ and $H_1 = 0$,
- or $t = t_1$ and $V(t, x) = \varphi(x)$.

The set of all relaxed states is denoted by $\mathscr{R}$.

From the HJBI it follows that

$$\mathscr{U}(t,x) = \{h \mid (t, x + Bh) \in \mathscr{R}, \\ \mathscr{V}^{-}(t, x + Bh) = \mathscr{V}^{-}(t,x) - \|h\|\}.$$

Definition (Approximating Motions)

Fix impulse times $t_0 \leq \tau_1 < \tau_2 < \cdots < \tau_K = t_1$.
The **approximating motion** $x(\cdot)$ is defined by

1. $x(t_0) = x_0$;
2. $\dot{x}(t) = A(t)x(t)$ on each open interval (τ_{j-1}, τ_j);
3. $x(\tau_j + 0) = x(\tau_j) + B(\tau_j)h_j$ at each impulse time τ_j with some vector $h_j \in \mathscr{U}(\tau_j, x(\tau_j))$ (possibly zero);
4. the open-loop control is

$$U(t) = \sum_{j=1}^{K} h_j \chi(t - t_j)$$

Definition (Closed-Loop Trajectory)

A pair $(x(\cdot), U(\cdot))$ is a **closed-loop trajectory** under feedback $\mathscr{U}(t, x)$, if it is a weak* limit of approximating motions $\{(x_k(\cdot), U_k(\cdot))\}_{k=1}^{\infty}$.

Any open-loop control $U(\cdot)$ from the Formal Definition and the corresponding trajectory $x(\cdot)$ are limits of approximating motions.

Space-time system (see for details Motta, Rampazzo. Space-Time Trajectories of Nonlinear System Driven by Ordinary and Impulsive Controls. Diff. & Int. Eqns V8, N2 (1995)):

$$\begin{cases} dx/dt = (A(t(s))x(s) + C(t(s))v(s)) \cdot u^t(s) + B(t(s))u^x(s) \\ dt/ds = u^t(s) \\ \mathscr{J}(u(\cdot)) = \max\limits_{v(\cdot)} \left\{ \int_0^S \|u^x(s)\|\, ds + \varphi(x(S)) \right\} \to \inf \\ t(0) = t_0, \quad t(S) = t_1 \end{cases}$$

Extended control $u(s) = (u^x(s), u^t(s)) \in \mathscr{B}_1 \times [0,1]$.

Extended feedback:

$$\mathscr{U}_{ST}(t,x) = \operatorname{conv} \begin{cases} (0,1), & h = 0; \\ (h,0), & h \neq 0 \end{cases} \quad \text{for} \quad h \in \mathscr{U}(t,x).$$

Closed-loop impulse control system is a **hybrid system**.
It is classified as a *continuous-controlled autonomous-switching hybrid system*. See Branicky, Borkar, Mitter. A Unified Framework for Hybrid Control... IEEE TAC V43, N1 (1998).

Continuous dynamics in $\mathscr{M} = \{(t,x) \mid H_1 = 0\}$:

$$\dot{x}(t) = A(t)x(t) + C(t)v(t), \quad (t,x) in \mathscr{M}.$$

Autonomous switching set $\mathscr{M}^C$:

$$x^+(t) = x(t) + Bh.$$

Vector h is such that $(t, x^+(t))$ is a relaxed state and

$$V(t, x(t) + B(t)h) = V(t, x(t)) + \|h\|$$

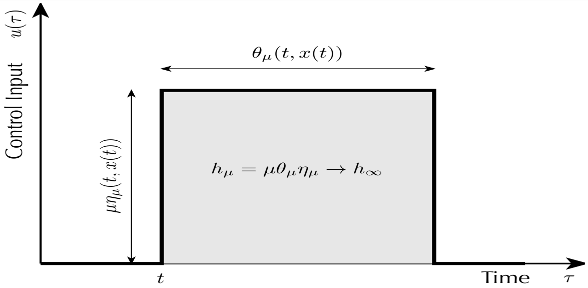
For further details see Kurzhanski, Tochilin. Impulse Controls in Models of Hybrid Systems. Diff. Eqns V45, N5 (2009).

Definition (Constructive Feedback)

A **constructive feedback control** is $\mathfrak{U} = \{\eta_\mu(t,x), \theta_\mu(t,x)\}$ s.t.

$$\eta_\mu(t,x) \in S_1 \cup \{0\} \qquad \eta_\mu(t,x) \underset{\mu\to\infty}{\to} \eta_\infty(t,x)$$

$$\theta_\mu(t,x) \geq 0 \qquad \mu\theta_\mu(t,x) \underset{\mu\to\infty}{\to} m_\infty(t,x)$$



Definition (Approximating Motion)

Fix $\mu > 0$ and times $t_0 = \tau_0 < \tau_1 < \ldots < \tau_s = t_1$.
An **approximating motion** is defined by

$$\tau_i^* = \tau_i \wedge (\tau_{i-1} + \theta_\mu(\tau_{i-1}, x_\Delta(\tau_{i-1})))$$
$$\dot{x}_\Delta(\tau) = A(\tau)x_\Delta(\tau) + \mu B(\tau)\eta_\mu(\tau_{i-1}, x_\Delta(\tau_{i-1})), \quad \tau_{i-1} < \tau < \tau_i^*$$
$$\dot{x}_\Delta(\tau) = A(\tau)x_\Delta(\tau), \quad \tau_i^* < \tau < \tau_i$$

Definition (Constructive Motion)

A *constructive motion* under feedback control $\mathfrak{U}$ is a pointwise limit point $x(\cdot)$ of approximating motions $x_\Delta(t)$ as $\mu \to \infty$ and $\sigma \to 0$.

Example

Example (A Scalar System)

$$dx = (1 - t^2)dU + v(t)dt, \quad t \in [-1, 1],$$

hard bound on disturbance $v(t) \in [-1, 1]$

$$\mathrm{Var}_{[-1,1]}\, U(\cdot) + 2|x(t_1 + 0)| \to \inf .$$

The value function is

$$\mathscr{V}^-(t, x) = \alpha(t)|x|, \quad \alpha(t) = \min\left(2, \min_{\tau \in [t,1]} \frac{1}{1 - \tau^2}\right).$$

Example

The Hamiltonians:

$$\mathscr{H}_1 = \begin{cases} \dfrac{tx}{1-t^2}, & \text{if } 0 \le t \le 1/\sqrt{2}, \\ 0, & \text{if } -1 \le t < 0, \text{ and } 1/\sqrt{2} < t \le 1. \end{cases}$$

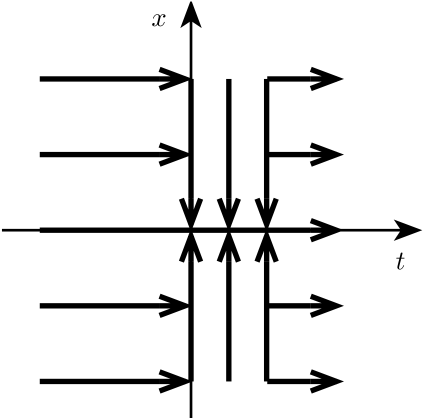
$$\mathscr{H}_2 = \begin{cases} t^2, & \text{if } -1 \le t < 0, \\ 2t^2 - 1, & \text{if } 1/\sqrt{2} < t \le 1, \\ 0, & \text{if } 0 \le t \le 1/\sqrt{2}. \end{cases}$$

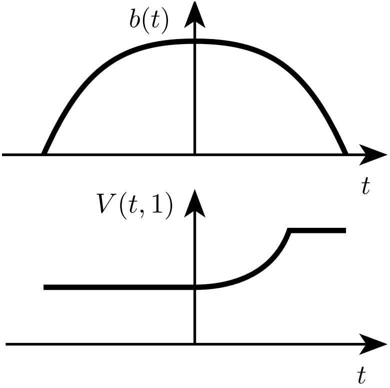
Feedback structure:

1. if $t < 0$ we have $\mathscr{H}_1 = 0$, $\mathscr{H}_2 \neq 0$ – do not apply control;
2. if $0 \le t \le 1/\sqrt{2}$, we have $\mathscr{H}_1 \neq 0$, $\mathscr{H}_2 = 0$ – apply an impulse control steering the system to the origin;
3. if $1/\sqrt{2} < t \le 1$, we have $\mathscr{H}_1 = 0$, $\mathscr{H}_2 \neq 0$, – do not apply control.

Example

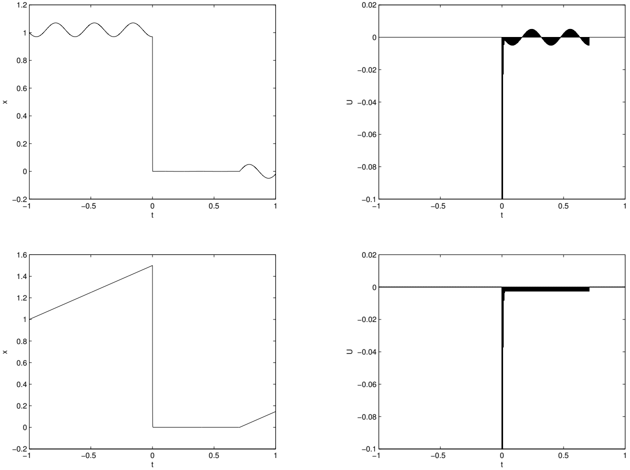
Feedback Control





Example

Realized Trajectories



Thank you for attention!