# Control Synthesis under Double Constraints

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# The System

The system

$$\begin{cases} \dot{x}(t) = B(t)u + v, \\ \dot{k}(t) = -\|u\|^2_{R(t)} \end{cases} \qquad (*)$$

is considered on a *fixed* time interval $T = [t_0, t_1]$.

*Control* $(u)$ is restricted by a *double constraint*

| hard bound | + | soft bound |
|---|---|---|
| $u \in \mathcal{P}(t)$ | | $k(t) \geqslant 0$ |
| | | $\Updownarrow$ |
| | | $\int_{t_0}^{t} \|u\|^2_{R(t)}\,\mathrm{d}t \leqslant k(t_0)$ |

*Disturbance* $(v)$ is subject to

hard bound

$$v \in \mathcal{Q}(t)$$

# Control classes

## $\mathcal{U}_{\mathrm{CL}}$ — *Closed-loop (feedback) strategies*

$$\mathcal{U}(t,x,k) : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R} \to \operatorname{conv} \mathbb{R}^n,$$

measurable in $t$ and u.s.c. in $(x,k)$,

$$\mathcal{U}(t,x,k) \subseteq \mathcal{P}(t),$$

$$\mathcal{U}(t,x,k) = \{0\} \quad \text{when} \quad k < 0.$$

These requirements are sufficient for the control to ensure the existence of solutions to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{k}(t) \end{pmatrix} \in \underbrace{\operatorname{conv} \left\{ \begin{pmatrix} B(t)u \\ -\|u\|^2_{R(t)} \end{pmatrix} \;\middle|\; u \in \mathcal{U}(t,x,k) \right\}}_{\mathcal{B}(t,\mathcal{U}(t,x,k))} + \mathcal{Q}(t) \tag{**}$$

and to obey the double constraint.

## $\mathcal{U}_{\mathrm{OL}} = \mathcal{U}_{\mathrm{OL}}(k_0)$ — *Open-loop controls*

$$u(t) : [t_0,t_1] \to \mathbb{R}^n, \qquad \text{measurable in } t,$$

$$u(t) \in \mathcal{P}(t), \qquad \int_{t_0}^{t_1} \|u\|^2_{R(t)} \, \mathrm{d}t \leqslant k_0.$$

# Set Cross-Sections

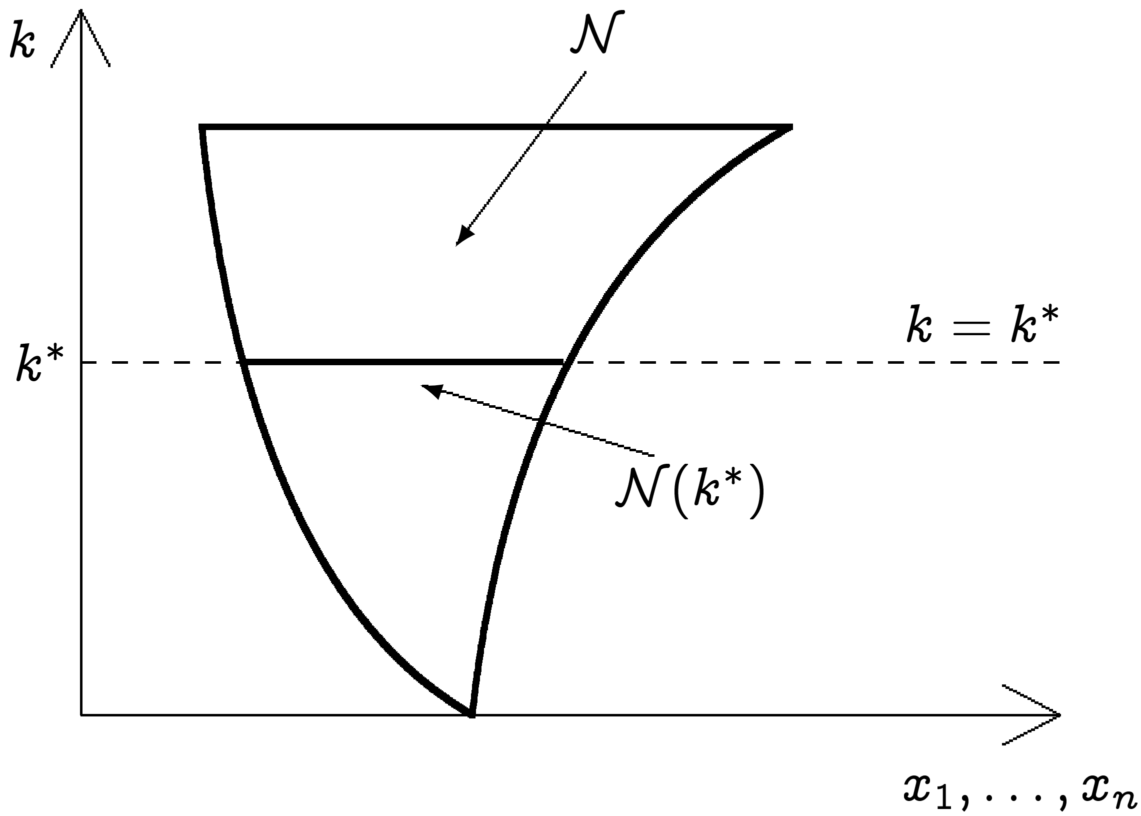
Let $\mathcal{N} \subseteq \mathbb{R}^n \times \mathbb{R}$ be a set of $(x, k)$ pairs.

$$\mathcal{N}(k) = \{\, x \in \mathbb{R}^n \mid (x, k) \in \mathcal{N} \}$$

is a <u>*cross-section*</u> of $\mathcal{N}$ at level $k$.

$$\mathcal{N} = \{\, (x, k) \mid x \in \mathcal{N}(k) \}$$

- $\star$ $\mathcal{N}$ convex $\Longrightarrow$ $\mathcal{N}(k)$ convex
- $\star$ $\mathcal{N}(k)$ convex $\Longrightarrow$ $\mathcal{N}$ quasi-convex
- $\star$ if $\mathcal{N}(k)$ are closed and locally bounded, then $\mathcal{N}$ closed $\iff$ $\mathcal{N}(\cdot)$ u.s.c.
- $\star$ $\mathcal{N} = \text{epi}\, f \Longrightarrow \mathcal{N}(k)$ is a level set of $f(\cdot)$ at level $k$.



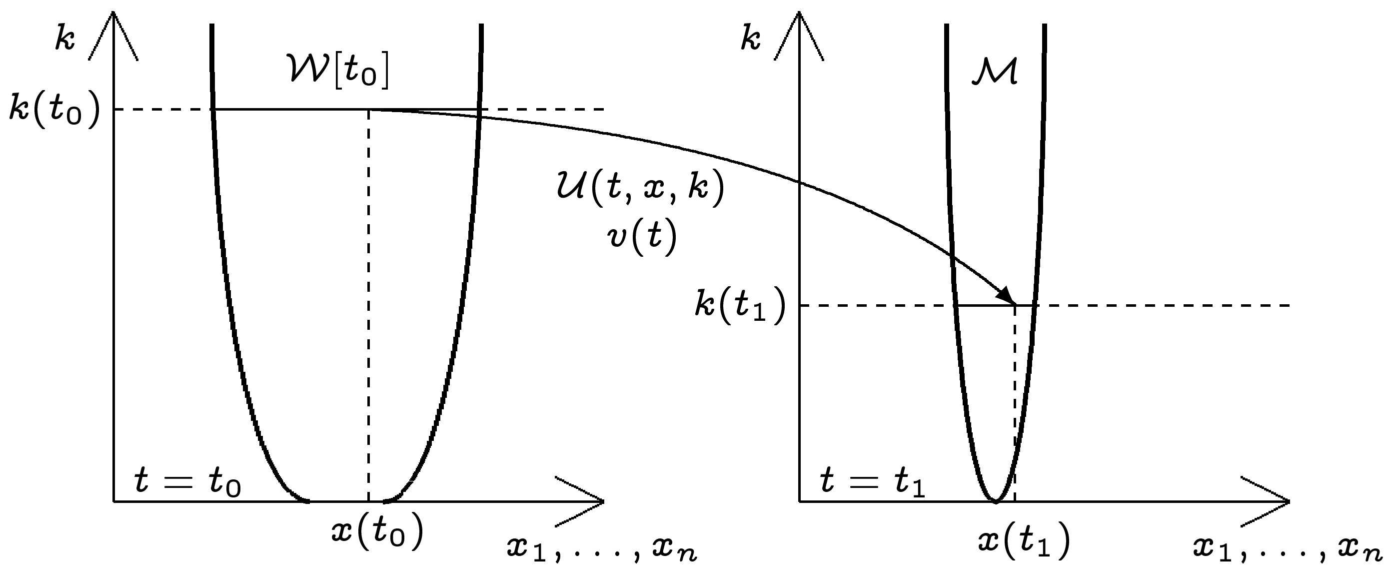
# The Problem

Let $\mathcal{M}$ be a non-empty *target set*, s.t.

1. $\mathcal{M}(k_1) \subseteq \mathcal{M}(k_2)$ if $k_1 \leqslant k_2$ (monotonicity);
2. $\mathcal{M}(k) = \varnothing$ for $k < 0$;
3. $\mathcal{M}(k)$ is Hausdorff-continuous when $\mathcal{M}(k) \neq \varnothing$;
4. $\mathcal{M}(k) \in \operatorname{conv} \mathbb{R}^n$.

**Problem.** Specify a *solvability domain* $\mathcal{W}[t] \subseteq \mathbb{R}^{n+1}$ and a feedback control strategy $\mathcal{U}(t, x, k) \in \mathcal{U}_{\mathrm{CL}}$ s.t. all the solutions of the differential inclusion $(**)$ starting from $(t, x(t)) \in \mathcal{W}[k(t), t]$ satisfy $x(t_1) \in \mathcal{M}(k(t_1))$.

(Ledyaev, 1985)



# Solution Outline

Problem is solved using the conjunction of the following concepts:

- ⋆ Pontryagin's alternated integral (Pontryagin, 1967; Pontryagin, 1980).
- ⋆ Krasovski's extremal construction (Krasovski, 1968; Krasovski and Subbotin, 1974).
- ⋆ "Nonsmooth" dynamic programming (Crandall and Lions, 1983; Subbotin, 1990).

Such combination was later considered by Kurzhanski (1999), Kurzhanski and Melnikov (2000) for problems with hard bounds on control and disturbance with the aim of introducing ellipsoidal methods (solving problem to the end).

# Alternated Integral

## Open-Loop Solvability Domains

*Max-Min Solvability Domain* $W^+(k, t, t_1; \mathcal{M}(\cdot))$ is the set of points $x \in \mathbb{R}^n$ s.t. for any admissible $v(\cdot)$ there exists $u(\cdot) \in \mathcal{U}_{\mathrm{OL}}(k)$ s.t. $x(t_1) \in \mathcal{M}(k(t_1))$.

(disturbance is known in advance)

*Min-Max Solvability Domain* $W^-(k, t, t_1; \mathcal{M}(\cdot))$ is the set of points $x \in \mathbb{R}^n$ s.t. there exists $u(\cdot) \in \mathcal{U}_{\mathrm{OL}}(k)$ for which $x(t_1) \in \mathcal{M}(k(t_1))$ for any admissible $v(\cdot)$.

(no information on disturbance)

Here $x(\cdot)$ is the trajectory of $(*)$ starting at point $(x, k)$ under control $u(\cdot)$.

# Alternated Integral

## Open-Loop Solvability Domains (cont)

$$W^+(k, t, t_1; \mathcal{M}(\cdot)) =$$

$$= \Big[ \bigcup_{0 \leqslant \gamma \leqslant k} \mathcal{M}(\gamma) - \mathcal{X}_{\mathrm{GI}}(t, t_1; k - \gamma) \Big] \dot{-} \int_t^{t_1} \mathcal{Q}(\tau)\, \mathrm{d}\tau,$$

$$W^-(k, t, t_1; \mathcal{M}(\cdot)) =$$

$$= \bigcup_{0 \leqslant \gamma \leqslant k} \Big[ \Big( \mathcal{M}(\gamma) \dot{-} \int_t^{t_1} \mathcal{Q}(\tau)\, \mathrm{d}\tau \Big) - \mathcal{X}_{\mathrm{GI}}(t, t_1; k - \gamma) \Big].$$

Here $\mathcal{X}_{\mathrm{GI}}$ is the *reach set under double constraint* (Daryin and Kurzhanski, 2001):

$$\mathcal{X}_{\mathrm{GI}}(t, t_1; k) = \left\{ \int_t^{t_1} B(\tau) u(\tau)\, \mathrm{d}\tau \;\middle|\; u(\cdot) \in \mathcal{U}_{\mathrm{OL}}(k) \right\}.$$

cf.: formulae for hard-bounds open-loop solvability sets:

$$W^+(t, t_1, \mathcal{M}) = \Big( \mathcal{M} - \int_t^{t_1} \mathcal{P}(\tau)\, \mathrm{d}\tau \Big) \dot{-} \int_t^{t_1} \mathcal{Q}(\tau)\, \mathrm{d}\tau,$$

$$W^-(t, t_1, \mathcal{M}) = \mathcal{M} \dot{-} \int_t^{t_1} \mathcal{Q}(\tau)\, \mathrm{d}\tau - \int_t^{t_1} \mathcal{P}(\tau)\, \mathrm{d}\tau.$$

# Alternated Integral

## Integral Sums

Let $\mathcal{T} = \{t = \tau_0, \tau_1, \dots, \tau_{m-1}, \tau_m = t_1\}$ be a partition of $[t, t_1]$ with $\sigma_i = \tau_i - \tau_{i-1} > 0$.

At $t = t_1$ set

$$W^+_{\mathcal{T}}[k, \tau_m] = W^-_{\mathcal{T}}[k, \tau_m] = \mathcal{M}(k),$$

and then at each point of $\mathcal{T}$

$$W^+_{\mathcal{T}}[k, \tau_{i-1}] = W^+(k, \tau_{i-1}, \tau_i; W^+_{\mathcal{T}}[\cdot, \tau_i]),$$
$$W^-_{\mathcal{T}}[k, \tau_{i-1}] = W^-(k, \tau_{i-1}, \tau_i; W^-_{\mathcal{T}}[\cdot, \tau_i]).$$

The sets

$$W^+_{\mathcal{T}}[k, \tau_0] = \mathcal{I}^+_{\mathcal{T}}(k, t, t_1; \mathcal{M}(\cdot)) = \mathcal{I}^+_{\mathcal{T}}[k, t],$$
$$W^-_{\mathcal{T}}[k, \tau_0] = \mathcal{I}^-_{\mathcal{T}}(k, t, t_1; \mathcal{M}(\cdot)) = \mathcal{I}^-_{\mathcal{T}}[k, t]$$

are *upper* and *lower alternated sums*, resp.

(Motion correction problem solvability domains)

# Alternated Integral

## Upper and Lower Integrals

*Assumption*: for every partition $\mathcal{T}$, $k \geqslant 0$ and $t \in [t_0, t_1]$ alternated sums $\mathcal{I}^+_{\mathcal{T}}[k,t]$ and $\mathcal{I}^-_{\mathcal{T}}[k,t]$ are convex.

If for some $k \geqslant 0$ there exists a Hausdorff limit $\mathcal{I}^+[k,t]$ of upper alternated sums

$$\lim_{\operatorname{diam}\mathcal{T}\to 0} h(\mathcal{I}^+_{\mathcal{T}}[k,t], \mathcal{I}^+[k,t]) = 0,$$

then it is referred to as *upper alternated integral*.

*Lower alternated integral* $\mathcal{I}^-[k,t]$ is defined the same way using lower alternated sums.

$$\mathcal{I}^+[k,t] = \bigcap_{\mathcal{T}} \mathcal{I}^+_{\mathcal{T}}[k,t],$$
$$\mathcal{I}^-[k,t] = \bigcup_{\mathcal{T}} \mathcal{I}^-_{\mathcal{T}}[k,t].$$

For all $t \in [t_0, t_1]$ and $k \geqslant 0$

$$\mathcal{I}^-[k,t] \subseteq \mathcal{W}[k,t] \subseteq \mathcal{I}^+[k,t].$$

## Alternated Integral

If both upper and lower integrals exist and they coincide, then $\mathcal{I}[k,t] = \mathcal{I}^+[k,t] = \mathcal{I}^-[k,t]$ is the *alternated integral*.

$$\mathcal{W}[k,t] = \mathcal{I}[k,t].$$

The class of mappings $\mathcal{M}(\cdot) \to \mathcal{I}(\cdot, t, \tau, \mathcal{M}(\cdot))$ is a two-parameter semigroup:

$$\mathcal{I}(k,t,t_1;\mathcal{M}(\cdot)) = \mathcal{I}(k,t,\tau,\mathcal{I}(\cdot,\tau,t_1;\mathcal{M}(\cdot))),$$
$$t_0 \leqslant t \leqslant \tau \leqslant t_1.$$

The same is true for mappings $\mathcal{I}^+$, $\mathcal{I}^-$, $\mathcal{W}$.

# Alternated Integral

## Convex Target Set

When target set $\mathcal{M}$ is convex, then the alternated integral is also convex and classical convergence theorems for hard bounds alternated integral (Ponomarev and Rozov, 1978) may be applied.

*Assumption A*: there exist continuous positive functions $\varkappa(t)$ and $r(t)$ s.t.

$$\forall \mathcal{T} \quad \mathcal{I}^{+}_{\mathcal{T}}[\varkappa(\tau_i), \tau_i] \supseteq B_{r(\tau_i)}.$$

*Assumption B*: there exist continuous positive functions $\varkappa(t)$, $r(t)$ and a number $\varepsilon > 0$ s.t.

$$\operatorname{diam} \mathcal{T} < \varepsilon \quad \Longrightarrow \quad \mathcal{I}^{-}_{\mathcal{T}}[\varkappa(\tau_i), \tau_i] \supseteq B_{r(\tau_i)}.$$

# Alternated Integral

## Convex Target Set (cont)

Denote

$$k_0^+(t) = \inf\left\{ k \;\middle|\; \forall \mathcal{T} \quad \mathcal{I}_{\mathcal{T}}^+(k, t, t_1, \mathcal{M}(\cdot)) \neq \varnothing \right\},$$

$$k_0^-(t) = \inf\left\{ k \;\middle|\; \exists \mathcal{T} \quad \mathcal{I}_{\mathcal{T}}^-(k, t, t_1, \mathcal{M}(\cdot)) \neq \varnothing \right\}.$$

Then

1. Under Assumption A, $\forall k \geqslant k_0^+(t)\ \exists \mathcal{I}^+[k,t]$;
2. Under Assumption B, $\forall k \geqslant k_0^-(t)\ \exists \mathcal{I}^-[k,t]$;
3. Under both assumptions $k_0^+(t) \equiv k_0^-(t) = k_0(t)$ and
   (a) $\mathcal{I}^+[k,t] = \mathcal{I}^-[k,t] = \mathcal{W}[k,t]$, $k > k_0(t)$;
   (b) $\mathcal{I}^+[k,t] = \mathcal{I}^-[k,t] = \varnothing$, $k < k_0(t)$;
   (c) $\mathcal{I}^-[k_0(t),t] \subseteq \mathcal{I}^+[k_0(t),t]$;
   (d) $\mathcal{I}^+[k,t] = \mathcal{W}[k,t]$, $\forall k \geqslant 0$.

# Dynamic Programming Approach

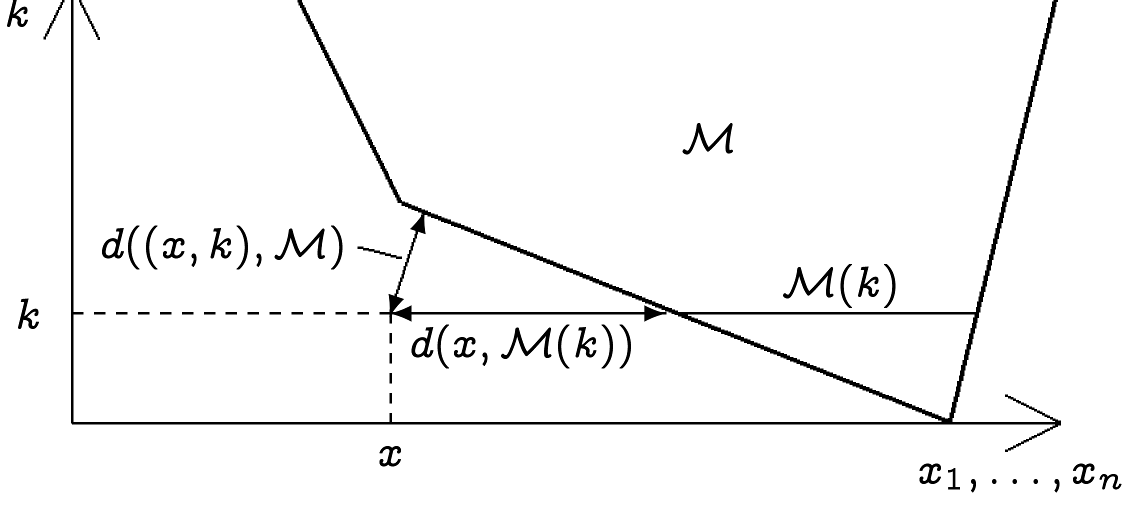
## The Value Function

$$V(t,x,k) = \inf_{\mathcal{U}\in\mathcal{U}_{\mathrm{CL}}} \sup_{\binom{x(\cdot)}{k(\cdot)}\in\mathcal{Z}_{\mathcal{U}}(\cdot)} d(x(t_1),\mathcal{M}(k(t_1))),$$

where $\mathcal{Z}_{\mathcal{U}}(\cdot)$ is the assembly of solutions to the differential inclusion $(**)$.

$$\mathcal{W}[k,t] = \{\, x \in \mathbb{R}^n \mid V(t,x,k) \leqslant 0\},$$

$$V(t,x,k) = \inf\{\, \mu \geqslant 0 \mid x \in \mathcal{W}(k,t,t_1;\mathcal{M}_\mu(\cdot))\},$$
$$V(t,x,k) \leqslant d(x,\mathcal{W}[k,t]).$$



# Dynamic Programming Approach

## Control Synthesis

The value function is a viscosity solution to the Hamilton–Jacobi–Bellman–Isaacs equation

$$V_t + \min_{u \in \mathcal{P}(t)} \max_{v \in \mathcal{Q}(t)} \left\{ \langle V_x, B(t)u + v \rangle - V_k \|u\|^2_{R(t)} \right\} = 0,$$
$$t_0 \leqslant t \leqslant t_1, \quad k \geqslant 0, \quad x \in \mathbb{R}^n$$

with boundary condition

$$V_t + \max_{v \in \mathcal{Q}(t)} \langle V_x, v \rangle \bigg|_{k=0} = 0, \quad t_0 \leqslant t \leqslant t_1, \quad x \in \mathbb{R}^n$$

and initial condition

$$V(t_1, x, k) = d(x, \mathcal{M}(k)), \quad k \geqslant 0, \quad x \in \mathbb{R}^n.$$

Optimal feedback strategy is

$$\mathcal{U}^*(t, x, k) = \operatorname*{Arg\,min}_{u \in \mathcal{P}(t)} \langle V_x, B(t)u \rangle - V_k \|u\|^2_{R(t)}.$$

# Evolution Equation

A multifunction $\mathcal{Z}[k,t] : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is said to be *weakly invariant* if the following inclusion holds:

$$\mathcal{Z}[k,t] \subseteq W^+(k,t,t+\sigma,\mathcal{Z}(\cdot,t+\sigma)) =$$

$$= \bigcup_{0\leqslant\gamma\leqslant k} (\mathcal{Z}[\gamma,t+\sigma]-\mathcal{X}_{\mathrm{GI}}(t,t+\sigma,k-\gamma)) \dot{-} \int_t^{t+\sigma} \mathcal{Q}(\tau)\,\mathrm{d}\tau.$$

Weak invariance is equivalent to the *u-stability* (Krasovski and Subbotin, 1974).

$\mathcal{Z}[k,t]$ is weakly invariant iff it is a solution to the *evolution equation*

$$\lim_{\sigma\downarrow 0} \sigma^{-1} h_+ \Bigg( \mathcal{Z}[k,t] + \sigma \mathcal{Q}(t),$$

$$\bigcup_{0\leqslant\gamma\leqslant k} (\mathcal{Z}[\gamma,t+\sigma] - \mathcal{X}_{\mathrm{GI}}(t,t+\sigma;k-\gamma)) \Bigg) = 0.$$

Solvability domain $\mathcal{W}[k,t]$ is the *maximum solution* to the evolution equation.