Impulse Control Inputs and the Theory of Fast Feedback Control

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Abstract: This paper deals with problems of impulse control which allow control inputs consisting not only of delta functions but also of their higher derivatives (impulses of higher order). The controls are sought for in the form of feedback strategies which leads to the application of respective generalized dynamic programming techniques, where the role of traditional Hamilton–Jacobi–Bellman equations is taken by respective variational inequalities of similar structure. Further proposed are physically realizable approximations which converge to these ideal solutions. Since the ideal solutions allow to transfer a controllable system from one given position to another in zero time, their approximations lead us to physically realizable “fast” controls with piecewise constant realizations. Such feedback control inputs are then compared with traditional bang-bang type strategies and turn out to be more robust. Computational schemes for related problems of reachability and control synthesis are further described with examples of damping oscillating systems of high order in minimal time being demonstrated.

Keywords: Switching stability and control, control design, optimization based controller synthesis, impulse feedback control, generalized dynamic programming, variational inequalities, reachability

1. INTRODUCTION

The present paper deals with problems of impulse control. It differs from other papers in this area due to the application of input controls which allow not only delta-functions, but also their higher derivatives (impulses of higher order). Besides that the controls are sought for in the form of closed-loop feedback strategies. We thus combine the properties of generalized functions with the dynamic nature of feedback control. This leads to the application of generalized dynamic programming techniques based on reducing the original problem to another one which is posed in the class of solutions which allow only delta-impulses. In such schemes the traditional Hamilton–Jacobi–Bellman equation is substituted by variational inequalities of similar structure. However these “ideal” solutions may not allow a physical realization. In order to make the solutions applicable we further introduce and array of physically realizable approximations with piecewise-constant control realizations. The approximations converge to the exact ideal solutions. Thus, since the ideal controls allow transition of the original controllable system from one given position to another in zero time, their physically realizable approximations — the so-called “fast” controls — allow to solve the same problem in finite time which is arbitrary small. The suggested impulse control synthesis is then compared with traditional bang-bang-type feedback strategies, turning out to be more robust in situations where the latter are not. Computational schemes for related problems of reachability and closed-loop control are thus conceived and examples of damping oscillations in high-order systems within small time are demonstrated.

2. IMPULSE CONTROL PROBLEMS

2.1 The Ordinary Impulse Controls

We start by recalling the classical impulse control problem (see Krasovski [1957], Neustadt [1964]). Consider a linear system

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t_0 \leq t \leq t_1. \]

Here \( x(t) \in \mathbb{R}^n \) is the state vector, and \( u(t) \) is the control input of form

\[ u(t) = \frac{dU(t)}{dt}, \]

where \( U(\cdot) \) is a function from the space \( BV([t_0, t_1]; \mathbb{R}^m) \) of \( m \)-vector functions of bounded variation, Riesz and Sz.-Nagy [1972]. Given matrix functions \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times m} \) are continuous.

The problem is to minimize the variation of control

\[ \text{Var} U(\cdot) \to \min \quad \text{subject to conditions } x(t_0) = x_0, \quad x(t_1 + 0) = x_1. \]

The initial time \( t_0 \) and terminal time \( t_1 \) are fixed. Here and below we assume that control \( U(t) \) and trajectories \( x(t) \) are left-continuous functions.

The problem (2) has been extensively studied in the class of open-loop controls. In particular, the minimum
optimal open-loop controls of such problem have the form

\[ u(t) = \sum_{i=1}^{\ell} \sum_{j=0}^{k} h_{i,j} \delta(t - \tau_i), \quad \ell \leq n, \]

and its norm (which replaces the variation in case of ordinary impulse controls) is defined as

\[ \rho^*[u] = \sum_{i=1}^{\ell} \gamma^*[\gamma_0^*[h_{i,0}], \ldots, \gamma_k^*[h_{i,k}]], \]

where \( \gamma, \gamma_0, \ldots, \gamma_k \) are norms in the corresponding finite-dimension spaces, \( \gamma^* \) denotes the adjoint norm.

The generalized problem may be reduced to a problem of type (2) as follows.

Assume that matrix functions \( A(t) \) and \( B(t) \) are \( k \) times continuously differentiable on the interval \( \alpha, \beta \supset [t_0, t_1] \).

Define functions \( L_j(t) \) by relations

\[ L_{0}(t) = B(t), \quad L_{j}(t) = A(t) L_{j-1}(t) - \frac{dL_{j-1}}{dt}, \quad j = 1, \ldots, k \]

and form the matrix \( \mathcal{B}(t) = (L_0(t) L_1(t) \cdots L_k(t)) \).

For a fixed \( \mathcal{B} \in \mathbb{R}^{m} \) the vector \( L_j(\tau) \mathcal{B} \) is equal to a jump of the trajectory of (1) at time \( \tau \) under control input \( u(t) = (-1)^j h(t) \delta(t - \tau) \).

Control inputs \( U(t) = (U_0^T(t) U_1^T(t) \cdots U_k^T(t))^T \) are chosen in the class \( BV([t_0, t_1]; \mathbb{R}^{m(k+1)}) \) of functions of bounded variations on the interval \( [t_0, t_1] \) with values in \( \mathbb{R}^{m} \).

This space is endowed with the following norm

\[ \text{Var}_{[t_0, t_1]} G^*[U(t)] = \sup_{t_1} G^*[U(t_{t+1}) - U(t_1)], \]

where \( G^* = (G^*_{[0, t_1]} G^*_{[t_1, t_2]}) \).

The corresponding generalized control is

\[ u(t) = \sum_{j=0}^{k} (-1)^{j+1} U_j \frac{dU_j}{dt}, \quad \rho^*[u] = \text{Var}_{[t_0, t_1]} G^*[U(t)]. \]

The generalized control problem is now equivalent to the problem of type (2) for system \( \dot{x}(t) = A(t)x(t) + \mathcal{B}(t)u(t) \).

3. THE FAST CONTROLS

3.1 The Ideal Zero-Time Controls

By applying the generalized control inputs system (1) may be controllable in zero time (i.e. \( t_0 = t_1 \)) even if the dimension of control \( m \) is less than the state space dimension \( n \). Indeed, this is the case when \( \text{im} \mathcal{B}(t_1) = \mathbb{R}^n \).

Then the minimal norm of the control is

\[ \min \rho^*[u] = \sup \{ (x, p) \mid G(\mathcal{B}(t_1)p) \leq 1 \}. \]

Here the supremum is taken over a bounded set since \( \ker \mathcal{B}(t_1) = \{0\} \).

However, such zero-time controls are idealistic and in real applications their physically realizable bounded approximations should be used. Such controls will allow a solution of the two-point boundary control problem in arbitrary small time.

3.2 The Realistic Fast Controls

Here we construct the so-called “fast” controls — bounded approximations of the control which solves the problem

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1. \( X(t, \tau) \) is the fundamental matrix of the homogeneous system, i.e. the solution to the matrix differential equation \( \frac{dX(t, \tau)}{dt} = A(t)X(t, \tau), \) \( X(\tau, \tau) = I \)
2. \( \chi(t) \) is the Heaviside step function, \( \delta(t) = \chi'(t) \) is Dirac’s delta-function.
in zero time. For sufficiently small $h > 0$ (such that the entire interval $[t_1, t_1 + k h]$ lies within $(\alpha, \beta)$) define scalar functions\footnote{ $\mathbf{1}_A(t)$ denotes the membership function of the set $A$, equal to 1 in $A$ and 0 in other points.}

$$
\Delta_h^j(t) = \frac{1}{h} \mathbf{1}_{[t, t+h]}(t), \quad \Delta_h^j(t) = \frac{1}{h} \Delta_h^{j-1}(t) - \Delta_h^{j-1}(t-h),
$$

$j = 1, \ldots, k$. These functions approximate the generalized functions $\delta^{(j)}(t)$. Consider controls of form $u(t) = \sum_{j=0}^k u_j \Delta_h^j(t - t_1)$, where $u_j \in \mathbb{R}^m$. Denote $U = (u_0^T u_1^T \cdots u_k^T)$.

$$
M_h^{(j)}(t) = \int_t^{t+kh} X(t + kh, \tau)B(\tau)\Delta_h^j(\tau - t) d\tau,
$$

and $M_h(t) = \sum_{r=0}^{N-1} M_h^{(r)}(t)$, then $x(t_1 + kh) = X(t_1 + kh, t_1) x(t_1) + M_h U$ and a natural analogue of the generalized control problem with $t_0 = t_1$ is the following finite-dimension optimization problem:

$$
\begin{align*}
\{G^*(U) &\to \text{inf}, \\
L_h(t_1) & U = x_1 - X(t + kh, t_1) x_0 = c.
\end{align*}
$$

(3)

Theorem 1. Suppose that rank $\mathcal{R}(t_1) = n$, then the problem (3) is solvable.

**Proof.** As $h \to 0^+$, the functions $\Delta_h^j(t)$ weakly converge to $\delta^{(j)}(t)$ in the space of distributions $D_{T_1}[\alpha, \beta]$. Thus

$$
M_h^{(j)}(t) \to (-1)^j \left. \frac{\partial X(t, \tau)B(\tau)}{\partial \tau} \right|_{\tau = t} = L_j(t),
$$

and $M_h(t) \to \mathcal{R}(t)$. Since by our assumption rank $\mathcal{R}(t_1) = n$, then for sufficiently small $h > 0$ we also have rank $\mathcal{R}_h(t_1) = n$ and the admissible controls do exist. Let $\tilde{U}$ be an arbitrary admissible control. The set $\{U \mid G^*(U), \mathcal{R}_h U = c\}$ is a compact, and the finite-dimension norm $G^*(U)$ is continuous. Therefore, the problem (3) has a solution.

Next we calculate the minimum value in problem (3):

$$
V_h = \min_{\mathcal{R}_h(\tau_1)U = G(\tau_1)} \max_{G(\tau_1)} \min_{\mathcal{R}_h(t_1)U = c} \langle q, U \rangle = \max_{\mathcal{R}_h(t_1)} \min_{G(\tau_1)} \langle q, U \rangle = \max \{ \langle (M_h^T(t_1))^p q, c \rangle \mid G(q) \leq 1 \}.
$$

Here $\oplus$ denotes a pseudo-inverse matrix (see Lancaster [1969]). We set $p = \langle (M_h^T(t_1))^p q, \rangle$, then

$$
V_h = \max \{ \langle p, c \rangle \mid G(M_h^T(t_1)p) \leq 1 \}.
$$

When $h \to 0$, we have $c \to x_1 - x_0$, $\mathcal{R}_h^T(\tau_1) \to \mathcal{R}^T(\tau_1)$. Therefore, $V_h \to \min_f \| u \|$.

### 3.3 The Norm of Fast Controls

Here we assume that $A(t) \equiv A$, $B(t) \equiv B$. According to Seidman and Yong [1997], the minimum variation of the impulse control is asymptotically

$$
N(\Delta) \sim \Delta_t^{-k}, \quad k = \min \{ j \mid x_0 \in R_j \},
$$

where $R_j = \text{im}(B \cdot AB \cdots A^j B)$.\footnote{Interpretation in terms of generalized controls. Suppose that $x_0 \in R_k \setminus R_{k-1}$. This means that the system is controllable in zero time by generalized controls of order (not less than) $k$ of the form $u(t) = \sum_{j=0}^k u_j \delta^{(j)}(t - t_1)$, where vectors $u_j$ solve the linear system $\sum_{j=0}^k L_j u_j = x_0$ with matrices $L_j = (-1)^j A^j B$. Note that $u_k \neq 0$ (otherwise controls of order $k - 1$ would suffice, which contradicts our assumption).}

It follows from (4) that for generalized controls (distributions) of order $r$

$$
N(\Delta) \sim \Delta_t^{-(k-r)}, \quad k = \min \{ j \mid x_0 \in R_j \}.
$$

(5)
Now if the control is a distribution of order \( r < k \),
then the terms with derivatives of order \( j \) greater than \( r \)
are replaced with finite-difference approximations of these
derivatives. The sum of coefficients of such approximations
are asymptotically of order \( \Delta t^{-(j-r)} \), which agrees with
(5).

A proof of (4). In (Seidman and Yong [1997]) the
estimate (4) follows from a general result for controls in
the class \( L_\rho \). Here we give a straightforward (and more
simple) proof of this estimate directly for scalar impulse
controls (\( B = b \in \mathbb{R}^n \)).

According to Neustadt [1964] the minimum norm of the
impulse control is attained on a control with at most \( n \)
impulses: \( u(t) = \sum_{j=1}^n u_j \delta(t - \tau_j) \), \( t_1 - \Delta t \leq \tau_j \leq t_1 \).
We assume that \( x_0 \in R \setminus R_{k-1} \), and without any loss of
generality, that \( t_1 = 0 \). Then the numbers \( u_j \) solve the
linear system \( \sum_{j=1}^n e^{-\tau_j} A b u_j = -x_0 \). Using the Taylor-
Maclaurin series for the matrix exponential, we have
\[
\sum_{j=1}^n \sum_{s=0}^k \frac{(-\tau_j)^s}{s!} A^s b u_j + O(\Delta t^{k+1})u = -x_0.
\]

Vectors \( A^s b, s = 0, \ldots, k \) are linearly independent, so \( x_0 \)
may be expressed through them (with some coefficients \( \xi_s \)):
\[
\sum_{j=1}^n \sum_{s=0}^k \frac{(-\tau_j)^s}{s!} A^s b u_j + O(\Delta t^{k+1})u = \sum_{s=0}^k \xi_s A^s b.
\]

Taking coefficients of \( A^s b \) on both sides, we come to the
linear system for \( u_j \):
\[
\sum_{j=1}^n \frac{(-\tau_j)^s}{\Delta t^s} u_j + O(\Delta t^{k+1-s})u = \frac{s! \xi}{\Delta t^s}, \quad s = 0, \ldots, k.
\]
The norm of the operator in the left-hand side is bounded
for small \( \Delta t \) uniformly over all choices of \( \tau_j \), the right-hand
side is of order \( \Delta t^{-k} \). Therefore, the control should have
a norm which asymptotically is not less than \( C \Delta t^{-k} \).

To prove the upper bound, we consider particular instants
of time \( \tau_j = -\frac{j}{n} \Delta t \):
\[
\sum_{j=1}^n \frac{j}{n} \frac{(-\tau_j)^s}{\Delta t^s} u_j + O(\Delta t^{k+1-s})u = \frac{s! \xi}{\Delta t^s}, \quad s = 0, \ldots, k.
\]
The operator \( M \) in the left-hand side is injective and
duly satisfies \( \|M u\| \geq C \|u\| \). From here it follows that
\( N(\Delta t) \leq C \Delta t^{-k} \).

The Realistic controls. Suppose that a physical realization of
the impulse \( u(t) = u_j \delta(t - t_j) \) is a “column”
\( u(t) = u_j h^{-1} 1_{[t_j, t_j+h]}(t) \). Then the respective “jump” of
the trajectory (actually not a jump, but a fast change in the
trajectory) is
\[
\Delta x(t_j) = e^{-hA} x(t_j + h) - x(t_j) = h^{-1} \int_{t_j}^{t_j+h} e^{(t_t - t)A} B u_j dt = F_h B u_j,
\]
where \( F_h = h^{-1} \int_0^h \delta^A dt \).

In the case of invertible matrix \( A \), \( F_h = h^{-1} A^{-1} (I - e^{-hA}) \).

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![Fig. 4. The high-order realistic impulse controls](image)

Fig. 4. The high-order realistic impulse controls

![Fig. 5. A realistic approximation of a generalized impulse control of order 3.](image)

Fig. 5. A realistic approximation of a generalized impulse control of order 3.

We see that the problem with physical realization of controls is equivalent to the problem of impulse control with matrix \( B \) replaced by \( F_h B \).

In a similar way we replace a generalized impulse \( u_j \delta^{(s)}(t - t_j) \) with its physical realization, namely, with a finite-difference approximation of \( s \)-th derivative of delta-function, \( \delta^{(s)}(t - t_j) \). The corresponding jump of \( x \) is
\[
\Delta x(t_j) = F_h^{(s)} B u_j, \quad \text{where } F_h^{(s)} = h^{-1} (1 - e^{-hA}) F_h^{(s-1)} \text{ with } F_h^{(0)} = F_h.
\]
We have \( F_h^{(s)} = h^{-s} (1 - e^{-hA})^s F_h \), and for invertible matrix \( A \), \( F_h^{(s)} = h^{-(s+1)} (1 - e^{-hA})^{s+1} A^{-1} \).

It follows that the problem with realistic controls — finite-
difference approximations of generalized functions of order
\( k \) — is equivalent to the problem of impulse control with matrix \( B \) replaced with \( (F_h^{(s)} B F_h^{(s+1)} B \cdots F_h^{(k)})B \).

In particular, the estimate 4 holds for this problem.

Note that \( F_h^{(s)} \to A^s \) as \( h \to 0 \).

Example 2. Fig. 4 shows a realization of the control for
Example 1 in the class of realistic impulses (\( h = 0.1 \)).
One may observe that this control corresponds to a finite-
difference approximation of the fifth derivative \( \delta^{(5)}(t - \tau) \).

Fig. 5 shows another realization of the control for Ex-
ample 1, now in the class of realistic controls of order \( r = 3 \).
4. THE CLOSED-LOOP CONTROL

4.1 The Dynamic Programming Approach

In order to formulate the principle of optimality, we consider the problem of minimizing a functional of Mayer–Bolza type:

$$\rho^* \left[ u \right] + \Phi(x(t_1 + 0)) \to \inf.$$  \hspace{1cm} (7)

The previously considered problems are particular cases of (7) if the function $\Phi$ is chosen as $^4 \Phi(x) = \mathcal{F}(x(x_i))$.

Denote the minimum value in problem (7) as $V(t_0, x_0) = V(t_0, x_0; t_1, \Phi(\cdot))$. The principle of optimality holds in the form of a semi-group property

$$V(t_0, x_0; t_1, \Phi(\cdot)) = V(t_0, x_0; t_{\epsilon}, V(t_{\epsilon}; \cdot; t_1, \Phi(\cdot))),$$

where $t_0 \leq t_{\epsilon} \leq t_1$.

The value function $V(t, x)$ satisfies a quasi-variational inequality (Bensoussan and Lions [1982]; Daryin, Kurzhanski and Seleznev [2005]) in the points of non-smoothness of the value function and its inner products stand for the respective directional derivatives

$$\min \{ H_1(t, x, V_x), H_2(t, x, V_x) \} = 0, \hspace{1cm} (8)$$

$$H_1(t, x, \xi_t, \xi_x) = \xi_t + \langle \xi_x, A(t)x \rangle,$$

$$H_2(t, x, \xi_t, \xi_x) = \min_{u \in \mathbb{R}^{m+1}} \left\{ \langle \xi_t, \mathcal{B}(t)u \rangle + 1 \mid G^*(u) \leq 1 \right\} =$$

$$1 - G(\mathcal{B}^T(t)V_x) = 1 - \gamma_1^T(t)V_x = \ldots = \gamma_k^T(t)V_x.$$

Here the Hamiltonian $H_1$ corresponds to motion without control ($dU = 0$), and $H_2$ accounts for the jumps generated by control impulses. Therefore, equation (8) may be interpreted as follows: either $H_1 = 0$ and in the respective on-line position one may choose zero control (no jump), or $H_2 = 0$ and the control will have an impulse.

4.2 A Formalization of the Closed-Loop System

Although a feedback control strategy may be formally derived from the Hamilton–Jacobi–Bellman equation, it is not clear what a closed-loop system would be under such control. A possible formalization for ordinary impulse controls is based on the double constraint approach from (Daryin, Kurzhanski and Seleznev [2005]).

Definition 1. The pair of functions $\mathcal{U} = \{ u(t, x; \mu), \theta(t, x; \mu) \}$ (“magnitude” and “duration”), such that

$$u(t, x; \mu) \in S_1 \cup \{0\}, \hspace{1cm} u(t, x; \mu) \xrightarrow{\mu \to \infty} u_\infty(t, x),$$

$$\theta(t, x; \mu) \geq 0, \hspace{1cm} \theta(t, x; \mu) \xrightarrow{\mu \to \infty} \theta_\infty(t, x),$$

is called the impulse feedback control strategy.

The component $u(t, x)$ is the direction of the control impulse which is issued on interval $[t, t + \theta(t, x)]$. Note that as $\mu \to \infty$, we have $\theta \to 0$ and in the limit one has a delta-function as control.

$^4$ By $\mathcal{F}(x|A)$ we denote the indicator function of a set $A$, which is zero on this set and equal to infinity elsewhere.

Fig. 6. Comparison of bang-bang controls vs impulse controls in $L_1$ norm

Definition 2. Fix a control strategy $\mathcal{U}$, number $\mu > 0$ and a partition $t_0 = t_0 < t_1 < \ldots < t_k = t_1$ of interval $[t_0, t_1]$. An approximating motion of the system (1) is the solution to the differential equation

$$\tau_i^* = \tau_i \wedge \theta(\tau_i-1, x\Delta(\tau_i-1); \mu),$$

$$x_\Delta(\tau) = \mu B(t)u(\tau_i-1, x\Delta(\tau_i-1); \mu), \hspace{0.5cm} \tau_i-1 < \tau < \tau_i^*,$$

$$x_\Delta(\tau_i) = x_\Delta(\tau_i^*).$$

Number $\sigma = \max\{\tau_i - \tau_i-1\}$ is the diameter of the partition.

Definition 3. A constructive motion of system (1) under feedback control $\mathcal{U}$ is a piecewise continuous function $x(t)$, which is the pointwise (weak) limit of approximating motions $x_\Delta(t)$ as $\mu \to 0$ and $\sigma \to 0$.

4.3 Impulses vs Bang-Bang Controls

In Fig. 6 we compare bang-bang controls from Pontryagin’s Maximum Principle with impulse controls in terms of the $L_1$ norm. We observe that the norm of impulse controls is twice lower than that of bang-bang controls.

As the control time, $t_1 - t_0$, decreases, the norm of the impulse control grows, and so does the minimum possible amplitude of bang-bang controls. Consider such a bang-bang feedback control with minimum amplitude $\mu$. Suppose that the switching of bang-bang control occurs after each $\theta$ time units ($\theta \ll t_1 - t_0$). Then the error $x(t_1) - x_1$ will be of order $\theta\mu$, which may be large unless $\theta \ll \mu^{-1}$. In other words, to get such feedback control functioning properly, one has to ensure very high sampling rate, which may be infeasible. The proposed formalization of impulse control is free of the indicated issue (although too low sampling rate may result in increase of the variation of control).

5. ON NUMERICAL SCHEMES

To avoid calculation of the exact value function, its upper and lower bounds may be described through quadratic approximations similar to those developed in ellipsoidal calculus (Kurzhanski and Vályi [1997], Kurzhanski and Varaiya [2000]). Such results may be then applied to the calculation of forward and backward reach sets under...
generalized impulse controls of the present paper. An ellipsoidal bound for reach sets under impulse controls is presented in detail in (Daryin and Kurzhanski [2007]).

6. CONCLUSION

In this paper we presented an approach to problems of generalized impulse feedback control in the class of inputs which allow not only delta-functions, but also their higher derivatives as controls. We also introduce the notion of realizable fast controls which solve the problems of this paper in arbitrary small “nano”-time. The suggested generalized inputs taken as virtual controls may be also relevant for systems with jumps and for on-line resets of the system structure and/or position, Kurzhanski [2006]. Further work is expected to include calculation and formalization of impulse feedback controls for systems under uncertainty, as well as the treatment of complex state constraints.

Appendix A. THE GENERALIZED IMPULSE CONTROL PROBLEM

Here the control $u(t)$ is chosen from the class $D^*_{k,m}[\alpha, \beta]$ of continuous linear functionals over the linear normed space $D_{k,m}[\alpha, \beta]$ (see Gelfand and Shilov [1964], Schwartz [1950]). The latter consists of $k$ times differentiable functions $\phi(t)$ : $[\alpha, \beta] \rightarrow \mathbb{R}^m$ with support contained in $[\alpha, \beta]$. The norm in $D_{k,m}$ is defined as

$$\rho[\phi] = \max_{t\in[\alpha,\beta]} |\gamma_0(\phi(t)), \gamma_1(\phi(t)), \ldots, \gamma_k(\phi^{(k)}(t))|,$$

where $\gamma_k, \gamma$ are finite-dimensional norms in spaces $\mathbb{R}^m$ and $\mathbb{R}^{k+1}$. The norm $\rho[\phi]$ determines its adjoint norm $\rho^*[u]$ in the space $D^*_{k,m}[\alpha, \beta]$. Hence the control is a distribution of order $k_u \leq k$. The trajectories of the system (1) are distributions from $D^*_{k-1,n}[\alpha, \beta]$.

The admissible controls $u(t)$ are distributions from $D^*_{k,m}[\alpha, \beta]$ for which there exists a distribution $x(t) \in D^*_{k-1,n}[\alpha, \beta]$ satisfying $x(t) = A(t)x(t) + B(t)u(t) + f^{(\alpha)} - f^{(\beta)}$ with support of $x(t)$ enclosed in $[t_0, t_1]$, where $\alpha < t_0 \leq t_1 < \beta$. Here $f^{(\alpha)}$ and $f^{(\beta)}$ are distributions from $D^*_{k,m}[\alpha, \beta]$, concentrated at points $t_0$ and $t_1$ respectively. These distributions may be interpreted as initial and final conditions for the trajectory $x(t)$ and may be represented as $f^{(\alpha)} = \sum_{j=0}^{k} \alpha_j \delta^{(j)}(t-t_0), f^{(\beta)} = \sum_{j=0}^{k} \beta_j \delta^{(j)}(t-t_1)$. Recall that any distribution $u \in D^*_{k,m}[\alpha, \beta]$ may be written as (see Gelfand and Shilov [1964])

$$\langle u, \phi \rangle = \sum_{j=0}^{k} \int_{\alpha}^{\beta} \frac{d^j \phi}{d^j t} dU_j(t),$$

where $U_j$ are functions of bounded variation on $[\alpha, \beta]$, taking values in $\mathbb{R}^m$ and constant when $\alpha \leq t \leq t_0$, $t_1 < t < \beta$.

A rigorous formulation of problem 7 sounds as follows: for a given distribution $f^{(\alpha)}$ and a time interval $[t_0, t_1]$ find a distribution $f^{(\beta)}$ and an admissible control $u(t) \in D^*_{k,m}[\alpha, \beta]$ minimizing the functional $J(u, f^{(\beta)}) = \rho^*[u] + \phi(f^{(\beta)})$. Here $\phi(f)$ is a convex, bounded from below terminal function.

REFERENCES


