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Estimation of Reachability Sets for Large-Scale Uncertain Systems: from Theory to Computation

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Abstract— **The estimation of reachability sets for systems of high dimensions is a challenging issue due to its high computational complexity. For linear systems, an efficient way of calculating such estimates is to find their set-valued approximations provided by ellipsoidal calculus. The present paper deals with various aspects of such approach as applied to systems of high dimensions with unknown but bounded input disturbances. We present an innovative technique based on parallel computation that involves on-line mixing of ellipsoidal tubes found in parallel. This improves robustness of the ellipsoidal estimates. Finally discussed is an implementation of the algorithm intended for supercomputer clusters.**

I. Introduction

Among estimation problems for controlled systems one of the key issues is to find the forward and backward reachability sets. This is especially important for systems with uncertainty where the topic is less developed and for systems of large dimensions for which there are very few results.

The notations used in this paper are as follows: $\mathscr{B}_r(a)$ is a ball of radius r centered at a in the corresponding normed space; $\langle x, y\rangle$ stands for the dot product of vectors x and y; $\|x\| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm of vector x; $x \parallel y$ ($x \upuparrows y$) means that vectors x and y are collinear (respectively, directionally collinear); number $\rho(\ell \mid \mathscr{X}) = \sup_{x\in\mathscr{X}} \langle \ell, x\rangle$ is the value of the *support function* for set $\mathscr{X}$ in direction ℓ; number $h_+(\mathscr{X}, \mathscr{Y}) = \inf\{\varepsilon \geq 0 \mid \mathscr{X} \subseteq \mathscr{Y} + \mathscr{B}_\varepsilon(0)\}$ (or, equivalently, $\sup\{\rho(\ell \mid \mathscr{X}) - \rho(\ell \mid \mathscr{Y}) \mid \|\ell\| \leq 1\}$) defines the Hausdorff semidistance between closed sets $\mathscr{X}$ and $\mathscr{Y}$; $\operatorname{conv} \mathscr{X}$ stands for the convex hull of set $\mathscr{X}$.

The system considered is

$$\dot{x}(t) = A(t)x(t) + B(t)u + C(t)v(t), \quad t \in [t_0, t_1], \tag{1}$$

with $x(t) \in \mathbb{R}^n$ being the phase trajectory that starts at given time t_0 as some $x^0 = x(t_0) \in \mathscr{X}^0$. Here u is the control and $v = v(t)$ is the disturbance. The unknown but bounded uncertain items x^0, u, v are confined to given convex compact sets $\mathscr{X}^0 \subseteq \mathbb{R}^n$, $\mathscr{P}(t) \subseteq \mathbb{R}^{n_u}$, $\mathscr{Q}(t) \subseteq \mathbb{R}^{n_v}$ accordingly. The control u may be treated in the class of either open-loop functions $u(t)$ or closed-loop (feedback) strategies $u = \mathscr{U}(t, x)$ that ensure existence of solution to (1). By $G(t,\tau)$ we denote the fundamental matrix (Green's function) of the system (1) (solution to Cauchy's problem $\partial G/\partial t = A(t)G(t,\tau)$, $G(\tau,\tau) = I$).

The reachability set $\mathscr{X}[\tau]$ of system (1) at time τ from position $\{t_0, \mathscr{X}^0\}$ is the set of all states x_τ reachable by some closed-loop control $u = \mathscr{U}(t,x)$ from some point $x^0 \in \mathscr{X}^0$, despite the unknown disturbance $v(\cdot)$. Set-valued function $\mathscr{X}[t]$, $t \geq t_0$ is *the reachability tube*.

The basic problem treated here is how to calculate ellipsoidal estimates for the reachability tube of system (1) under indicated set-membership uncertainties, [1], [2].

Note that solutions to the stated problem are crucial tools for dealing with **the problem of target feedback control** which is to find a feedback control strategy $\mathscr{U}(t,x)$ that steers system (1) from given position $\{t_0, \mathscr{X}^0\}$ to given target set $\mathscr{M}$, so that $x(t_1) \in \mathscr{M}$ despite any admissible disturbance $v(\cdot)$. For (1) this may be solved in terms of the *backward reachability* (solvability) *tube* $\mathscr{W}[t]$ for (1) which are defined similarly to $\mathscr{X}[t]$, but in backward time. Exact calculation of $\mathscr{W}[t]$ for systems of higher dimensions is computationally too cumbersome, hence we use $\mathscr{W}^-[t]$ – its internal ellipsoidal approximations [1], [2]

With $\mathscr{W}[t]$ given, the control strategy is

$$\mathscr{U}^*(t,x) = \begin{cases} -\frac{P(t)B^T(t)p}{\langle B^T(t)p, P(t)B^T(t)p\rangle^{\frac{1}{2}}}, & B^T(t)p \neq 0; \\ \mathscr{P}(t), & B^T(t)p = 0; \end{cases}$$

where the vector $p = p(t,x) = \partial V/\partial x$, $V(t,x) = d(G(t_1,t)x, G(t_1,t)\mathscr{W}[t])$, is the direction of the shortest path from x to $\mathscr{W}[t]$ (see [1]).

Recall that an ellipsoid $\mathscr{E}(x, X) \subseteq \mathbb{R}^k$ with center $x \in \mathbb{R}^k$ and non-negative definite configuration matrix $X \in \mathbb{R}^{k\times k}$ is a convex compact set with support function $\rho(\ell \mid \mathscr{E}(x,X)) = \langle \ell, x\rangle + \langle \ell, X\ell\rangle^{\frac{1}{2}}$, see [3], [4].

Here we require that all sets in the problem are ellipsoids: $\mathscr{X}^0 = \mathscr{E}(x^0, X^0)$, $\mathscr{P}(t) = \mathscr{E}(p(t), P(t))$, $\mathscr{Q}(t) = \mathscr{E}(q(t), Q(t))$. Otherwise sets $\mathscr{P}(t)$ have to be approximated by their internal ellipsoids and set $\mathscr{Q}(t)$ by their externals [2]. Functions $p(t)$, $P(t)$, $q(t)$, $Q(t)$ for the constraints are assumed to be continuous.

An earlier result [2] is that the internal $\mathscr{X}^-[t] = \mathscr{E}(x^*(t), X^-(t))$ and external $\mathscr{X}^+[t] = \mathscr{E}(x^*(t), X^+(t))$ ellipsoidal estimates of the reachability tube $\mathscr{X}[t]$ for (1) are solutions to the following ODEs:

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)p(t) + C(t)q(t), \tag{2}$$

with initial condition $x^*(t_0) = x^0$, and

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$$\dot{X}^+(t) + A(t)X^+(t) + X^+(t)A^T(t) + \pi_+(t)X^+(t) + \pi_+^{-1}(t)B(t)P(t)B(t) - R_S\left[(X^+(t))^{\frac{1}{2}}S_+(t)(C(t)Q(t)C^T(t))^{\frac{1}{2}}\right] = 0, \quad (3)$$

$$\dot{X}^-(t) + A(t)X^-(t) + X^-(t)A^T(t) - R_S\left[(X^-(t))^{\frac{1}{2}}S_-(t)(B(t)P(t)B^T(t))^{\frac{1}{2}}\right] + \pi_-(t)X^-(t) + \pi_-^{-1}(t)C(t)Q(t)C(t) = 0, \quad (4)$$

with initial condition $X^{\pm}(t_0) = X^0$ and $R_S[Q] = Q + Q^T$.

In (3), the parameterizing function

$$\pi_+(t) = \left\langle s(t), B(t)P(t)B^T(t)s(t)\right\rangle^{\frac{1}{2}} / \left\langle s(t), X^+(t)s(t)\right\rangle^{\frac{1}{2}},$$

where $s(t)$ is the solution to the adjoint system

$$\dot{s}(t) = -A^T(t)s(t), \quad s(t_0) = \ell.$$

$S_+(t)$ is an orthogonal matrix that ensures vectors $S_+(t)(C(t)Q(t)C^T(t))^{\frac{1}{2}}s(t)$ and $(X^+(t))^{\frac{1}{2}}s(t)$ to be collinear. (With no disturbance v take $S_+ \equiv 0$).

In (4), $S_-(t)$ is an orthogonal matrix such that vectors $S_-(t)(B(t)P(t)B^T(t))^{\frac{1}{2}}s(t)$ and $(X^-(t))^{\frac{1}{2}}s(t)$ are collinear. In the absence of disturbance the two terms in (4) with $\pi(t)$ are omitted. Otherwise

$$\pi_-(t) = \left\langle s(t), C(t)Q(t)C^T(t)s(t)\right\rangle^{\frac{1}{2}} / \left\langle s(t), X^-(t)s(t)\right\rangle^{\frac{1}{2}}.$$

Ellipsoidal approximations (4), (3) are "tight" in the sense that they touch the exact reachability set in the direction of the adjoint vector $s(t)$:

$$\rho\left(s(t) \mid \mathscr{X}^+[t]\right) = \rho\left(s(t) \mid \mathscr{X}^-[t]\right) = \rho\left(s(t) \mid \mathscr{X}[t]\right).$$

The calculation of external estimates is less difficult than of the internals which are less tackled due to arising serious difficulties. However, the internals are crucial for many applications and these difficulties, which increase with dimension, have to be overcome. We therefore concentrate on internals.

An attempt to apply established approximation formulas from [2] to oscillating systems of high dimensions [5] with scalar control ($n_u = 1$) revealed that the matrix of ellipsoid $X^-[t]$ is ill-conditioned (namely, $n-1$ semi-axes of the ellipsoid are of length close to zero). This presents a serious issue for practical computations since

1) errors caused by numerical integration of approximation's ODE may cause matrix $X^-[t]$ to have negative eigenvalues, which is unacceptable;
2) disturbances acting in directions where $X^-[t]$ is degenerate lead to $X^-[t]$ collapsing quickly;
3) the only information provided by a degenerate ellipsoid with a single positive axis is the value of support function in the direction of that axis. However, this value may be obtained by simpler calculations (not involving the solution of a matrix ODE).

In section II we present an amendment for ellipsoidal approximation formulas from [2]. The idea is to calculate a set of ellipsoidal approximations which are then "mixed".

Another issue with high-dimension systems is its polynomially increasing computational load. Section III introduces a new efficient method for calculating matrices $S_{\pm}(t)$.

Finally, in section IV we discuss a parallelized software implementation of the presented formulas. The given algorithms have proved to be effective for systems of ODE's of dimension up to 500.

II. Regularizing the Estimates

Our numerical experiments with internal ellipsoidal estimates of the reachability tube for a high-order oscillating system [5] with a scalar control lead to the following conclusion. If the initial set $\mathscr{X}^0$ has small diameter, then the internal ellipsoidal estimates of the reach tube are close to degenerate. Namely, as time t increases, only one eigenvalue of matrix $X^-(t)$ grows. Other eigenvalues remain close to those of matrix X^0.

We start this section by demonstrating the cause of such degeneracy. To do this, we analyse the formula for the internal ellipsoidal estimate of the geometrical sum of two ellipsoids. After that we indicate how one can overcome this degeneracy by weakening the requirements on the tightness of estimates. Finally we extend this approach to calculating the internal estimates of the reachability tube.

A. *Degeneracy of Sum of Degenerate Ellipsoids*

Recall [2] the formula for an internal ellipsoidal estimate of the geometrical sum of m ellipsoids $\mathscr{E}(q_i, Q_i)$, $i = 1, m$, tight in direction $\ell \in \mathbb{R}^n$:

$$\mathscr{E}(q_1, Q_1) + \cdots + \mathscr{E}(q_m, Q_m) \supseteq \mathscr{E}(q, Q), \quad (5)$$
$$q = \sum_{i=1}^{m} q_i, \quad Q = R_P(Q_1^{\frac{1}{2}} + S_2 Q_2^{\frac{1}{2}} + \cdots + S_m Q_m^{\frac{1}{2}}),$$

where $R_P(Q) = Q^T Q$, and S_i are orthogonal matrices satisfying $Q_1^{\frac{1}{2}}\ell \upuparrows S_i Q_i^{\frac{1}{2}}\ell$. If any of the matrices Q_i is degenerate, relation (5) makes sense for directions ℓ such that $Q_i\ell \neq 0$, $i = 1, m$.

Theorem 1: Suppose $\operatorname{rank} Q_i = r_i$ and $Q_i\ell \neq 0$, $i = 1, m$. Then the rank of matrix Q calculated by (5) is limited by $r_1 + \cdots + r_m - (m-1)$.

Proof: Denote column spaces of matrices $Q_1^{\frac{1}{2}}$ and $S_i Q_i^{\frac{1}{2}}$ by L_1 and L_i respectively, $i = \overline{2, m}$. Then $\operatorname{im} Q \subseteq \sum_{i=1}^{m} L_i$. By definition of matrices S_i the non-zero vector $e_1 = Q_1^{\frac{1}{2}}\ell$ belongs to all subspaces L_i. Therefore the total number of linearly independent vectors in $\operatorname{im} Q$ cannot be greater than $r_1 + (r_2 - 1) + \cdots + (r_m - 1)$. ∎

Corollary 1: The set of rank 1 matrices is closed with respect to (5).

Corollary 2: Consider system (1) with scalar control ($\operatorname{rank} B(t)P(t)P^T(t) \equiv 1$) without uncertainty, with initial set being a point or an interval ($\operatorname{rank} X^0 \leqslant 1$). Then the ellipsoidal estimate (2), (4) for such system will also be a point or an interval at each fixed time ($\operatorname{rank} X^-(t) \leqslant 1$).

Proof: This statement follows from the fact that all Euler approximations to (4) will be of rank 1. ∎

Remark 1: The degeneracy of estimates is the result of their tightness (the requirement that estimates touch the exact set). This is not a property of a particular formula (5). Indeed, if $\operatorname{rank} Q_i = 1$, then the summed ellipsoids may be presented as $\mathscr{E}_i = \mathscr{E}(q_i, Q_i) = \operatorname{conv}\{q_i \pm a_i\}$, where vectors a_i are the only non-zero semi-axes of these ellipsoids. The exact sum $\mathscr{E}_1 + \dots + \mathscr{E}_m$ is a polyhedron $\operatorname{conv}\{q_1 + \dots + q_m \pm a_1 \pm \dots \pm a_m\}$. Along almost all directions ℓ the only tight approximation will be one of the diagonals of this polyhedron. Formula (5) describes precisely those diagonals.

Remark 2: For some choices of orthogonal matrix S the dimension of the internal ellipsoidal estimate may be strictly less than the dimension of ellipsoids being added. For example, if $Q_1 = Q_2 = I$, $\ell = e_1$, $S = \operatorname{diag}\{1, -1, -1, \dots, -1\}$, then (5) gives $Q = \operatorname{diag}\{1, 0, 0, \dots, 0\}$.

B. Regularizing the Estimate for the Sum of Degenerate Ellipsoids

Here we show how several degenerate estimates may be combined (mixed) to get an estimate of higher dimension. This approach is based on the formula for the internal ellipsoidal estimate for the convex hull of the union of ellipsoids [1].

Lemma 2: If ellipsoids $\mathscr{E}_i^- = \mathscr{E}(q, Q_i^-)$, $i = \overline{1,m}$, are internal estimates of a convex set $\mathscr{X}$, then ellipsoid

$$\mathscr{E}_\alpha^- = \mathscr{E}(q, Q_\alpha^-), \quad Q_\alpha^- = \sum_{i=1}^m \alpha_i Q_i^-, \quad \alpha_i \geqslant 0, \quad \sum_{i=1}^m \alpha_i = 1, \tag{6}$$

will also be an internal estimate of $\mathscr{X}$.

Proof: Since $\sqrt{t}$ is concave, we have

$$\rho(\ell \mid \mathscr{E}_\alpha^-) = \langle q, \ell\rangle + \left(\sum_{i=1}^m \alpha_i \langle \ell, Q_i^- \ell\rangle\right)^{\frac{1}{2}} \leq$$
$$\sum_{i=1}^m \alpha_i (\langle q_i, \ell\rangle + \langle \ell, Q_i^- \ell\rangle^{\frac{1}{2}}) \leqslant \max_{i=\overline{1,m}} \rho(\ell \mid \mathscr{E}_i^-), \tag{7}$$

hence $\mathscr{E}_\alpha^- \subseteq \operatorname{conv} \bigcup_{i=1}^m \mathscr{E}_i^- \subseteq \mathscr{X}$. ■

Theorem 3: Suppose $q = 0$ and the dimension of $\mathscr{L}$ (the linear hull of $\mathscr{E}_i^-$) is r. Then if $\alpha_i > 0$, $i = \overline{1,m}$, the following equality holds: $\operatorname{im} Q_\alpha^- = \mathscr{L}$. In particular, the matrix Q_α^- is of rank r.

Proof: The statement of this theorem means that

$$\operatorname{im}(\alpha_1 Q_1^- + \dots + \alpha_m Q_m^-) = \operatorname{im} Q_1^- + \dots + \operatorname{im} Q_m^-.$$

The latter, due to the symmetry of matrices Q_i^-, is equivalent to

$$\ker(\alpha_1 Q_1^- + \dots + \alpha_m Q_m^-) = \ker Q_1^- \cap \dots \cap \ker Q_m^-.$$

The inclusion of the right-hand side into the left is obvious. If $x \in \ker(\alpha_1 Q_1^- + \dots + \alpha_m Q_m^-)$, then taking the dot product of equality $\alpha_1 Q_1^- x + \dots + \alpha_m Q_m^- x = 0$ by x, we get $\alpha_1 \langle x, Q_1^- x\rangle + \dots + \alpha_m \langle x, Q_m^- x\rangle = 0$. Since matrices Q_i^- are non-negative definite and α_i is positive, we have $\langle x, Q_i^- x\rangle = 0$. Therefore $x \in \ker Q_i^x$. ■

Note some properties of estimates $\mathscr{E}_\alpha^-$:

1) Suppose that ellipsoidal approximation $\mathscr{E}_1^-$ is tight in direction ℓ, i.e. $\rho(\ell \mid \mathscr{E}_1^-) = \rho(\ell \mid \mathscr{X})$. Let us estimate the difference between its support function and that of ellipsoid $\mathscr{E}_\alpha^-$ in the same direction:

$$\rho(\ell \mid \mathscr{E}_\alpha^-) \geqslant \langle q, \ell\rangle + (\alpha_1 \langle \ell, Q_1 \ell\rangle)^{\frac{1}{2}} =$$
$$\langle q, \ell\rangle + \langle \ell, Q_1 \ell\rangle^{\frac{1}{2}} \sqrt{1 - (1 - \alpha_1)} =$$
$$= \rho(\ell \mid \mathscr{X}) - \tfrac{1}{2}(1 - \alpha_1) \langle \ell, Q_1 \ell\rangle^{\frac{1}{2}} + O((1 - \alpha_1)^2).$$

Therefore, the closer α_1 is to 1, the closer are ellipsoids $\mathscr{E}_\alpha^-$ to the tight approximation in direction ℓ.

2) If all the ellipsoids $\mathscr{E}_i^-$, $i = \overline{1,m}$, are tight approximations of $\mathscr{X}$ in the same direction ℓ, then in (7) an equality holds and the ellipsoid $\mathscr{E}_\alpha^-$ is also a tight approximation of $\mathscr{X}$ in direction ℓ.

Example 1: Figure 1 shows ellipsoidal estimates of the sum of two degenerate ellipsoids. Original ellipsoids $\mathscr{E}_i$ are two line segments (thick dotted line), and their sum is a parallelogram (thin dotted line). Due to Theorem 1, tight approximations of the sum are also degenerate ellipsoids (line segments) shown by thick solid line. Regularized approximations with $\alpha_1 = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ are presented with thin solid lines. They are non-degenerate (Theorem 3) and touch the parallelogram (i.e. they are tight in direction of normals to parallelogram sides — property 2). Besides that, for $\alpha = \frac{1}{10}$ and $\frac{9}{10}$ thanks to property 1 support functions of estimates are close to support function of the parallelogram in corresponding directions.

Fig. 2 shows internal ellipsoidal approximations of solvability set, as calculated by (8), for an oscillating system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$ on time interval $[0, \pi]$ with parameter $\gamma = \frac{1}{20}$ (left) and $\frac{1}{2}$ (right). Exact (tight) approximations are degenerate ellipsoids (shown with thick lines).

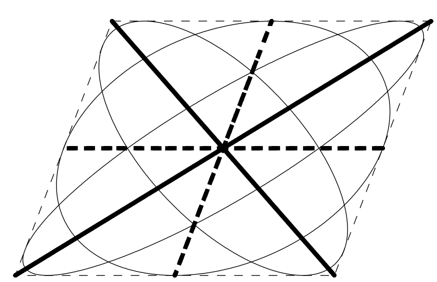

Fig. 1. Internal ellipsoidal approximations of sum of two ellipsoids

C. Regularizing the Estimate of the Reachability Tube

We choose in (6) the following values of parameters α: $\alpha_1 = 1 - \sigma\gamma + \sigma\gamma\beta_i$, $\alpha_i = \sigma\gamma\beta_i$, $i = \overline{2,m}$, where $\beta_i \geqslant 0$, $\sum_{i=1}^m \beta_i = 1$, $\gamma \geqslant 0$. Here σ is sufficiently small, such that $\alpha_1 > 0$. Then

$$\sigma^{-1}(Q_\alpha^- - Q_1^-) = \gamma\left(\sum_{i=1}^m \beta_i Q_i^- - Q_1^-\right).$$

We use this result to mix m ellipsoidal estimate of the reachability tube:

$$\dot{X}_i^-(t) + A(t)X_i^-(t) + X_i^-(t)A^T(t) - \\ - R_S((X_i^-(t))^{\frac{1}{2}}S_i(t)(B(t)P(t)B^T(t))^{\frac{1}{2}}) + \\ \pi_i(t)X_i^-(t) + \pi_i^{-1}(t)C(t)Q(t)C(t) + \\ \gamma\left(\sum_{j=1}^{m}\beta_{ij}X_j(t) - X_i^-(t)\right), \quad X_i^-(t_0) = X^0; \quad (8)$$

Here $\beta_{ij} \geqslant 0$, $\sum_{j=1}^m \beta_{ij} = 1$, $i = \overline{1,m}$; $\gamma \geqslant 0$; $S_i(t)$ are arbitrary orthogonal matrices such that vectors $S_i(t)(B(t)P(t)B^T(t))^{\frac{1}{2}}s_i(t)$ and $(X_i^-(t))^{\frac{1}{2}}s_i(t)$ are directionally collinear; functions $s_i(t)$ are the solutions to adjoint system with different initial conditions: $\dot{s}_i(t) = -A^T(t)s_i(t)$, $s_i(t_0) = \ell_i$, and in the presence of disturbances $\pi_i(t) = \left\langle s_i(t), C(t)Q(t)C^T(t)s_i(t)\right\rangle^{\frac{1}{2}} / \langle s_i(t), X_i^-(t)s_i(t)\rangle^{\frac{1}{2}}$.

Remark 3: Parameter $\gamma \geqslant 0$ controls how fast the approximations are "mixed" – the higher it is, the greater is the impact of mixing. Parameters $\beta_{ij} \geqslant 0$ control the configuration of mixing (which ellipsoids are added to the given one).

Remark 4: The choice of identical coefficients $\beta_{ij} = \hat{\beta}_j$ (in particular, $\beta_{ij} = \frac{1}{m}$) reduces the number of operations, since in this case the sum $\sum_{j=1}^m \beta_{ij}X_j^-(t) = \sum_{j=1}^m \hat{\beta}_j X_j^-(t)$ does not depend on j and is calculated only once for each time step.

Theorem 4: Suppose that solutions $X_i^-(t)$ to equation (8) are extendible to interval $[t_0, t_1]$, and are positive definite matrices on this interval. Then the set-value function

$$\mathscr{X}^-[t] = \operatorname{conv}\bigcup_{i=1}^{m}\mathscr{X}_i^-[t] = \operatorname{conv}\bigcup_{i=1}^{m}\mathscr{E}(x^*(t), X_i^-(t))$$

satisfies for $t \in [t_0, t_1]$ the funnel equation

$$\lim_{\sigma\to 0+}\sigma^{-1}h_+((I - \sigma A(t))\mathscr{X}^-[t+\sigma] - \sigma C(t)\mathscr{Q}(t), \\ \mathscr{X}^-[t] + \sigma B(t)\mathscr{P}(t)) = 0$$

with initial condition $\mathscr{X}^-[t_0] \subseteq \mathscr{X}^0$.

Proof: Without loss of generality we consider $p(t) \equiv 0$, $q(t) \equiv 0$ and therefore, $x^*(t) \equiv 0$. Then, using the change of variables $z(t) = G(t_0, t)x(t)$, we come to system (1) with $A(t) \equiv 0$. Besides that, for shorter notation we re-denote matrices $B(t)P(t)B^T(t)$ and $C(t)Q(t)C^T(t)$ as $P(t)$ and $Q(t)$ respectively.

The support function of the reachability set $\mathscr{X}^-[t]$ is $\rho(\ell \mid \mathscr{X}^-[t]) = \max\left\{\langle \ell, X_i^-(t)\ell\rangle^{\frac{1}{2}} \;\middle|\; i = 1, \ldots, m\right\}$. Let $\sigma > 0$ be sufficiently small, such that for $\delta \in [0, \sigma]$ the maximum for given direction ℓ is achieved on the same $i = i_0$, i.e. $\rho(\ell \mid \mathscr{X}^-[\tau]) = \left\langle \ell, X_{i_0}^-(\tau)\ell\right\rangle^{\frac{1}{2}}$.

Assuming $\|\ell\| = 1$, the estimate for the support function of $\mathscr{X}^-[t-\sigma]$ is

$$\rho(\ell \mid \mathscr{X}^-[t-\sigma]) = \left\langle \ell, X_{i_0}^-(t-\sigma)\ell\right\rangle^{\frac{1}{2}} = \\ \left\langle \ell, (X_{i_0}^-(t) - \sigma\dot{X}_{i_0}^-(t) + o(\sigma))\ell\right\rangle^{\frac{1}{2}} = \\ = \left\langle \ell, X_{i_0}^-(t)\ell\right\rangle^{\frac{1}{2}} - \frac{\sigma}{2}\left\langle \ell, X_{i_0}^-(t)\ell\right\rangle^{-\frac{1}{2}}\left\langle \ell, \dot{X}_{i_0}^-(t)\ell\right\rangle + o(\sigma).$$

Omitting the dependence on t, we further estimate

$$\left\langle \ell, \dot{X}_{i_0}^-\ell\right\rangle = -\langle \ell, (X_{i_0}^-)^{\frac{1}{2}}SP^{\frac{1}{2}}\ell\rangle - \langle \ell, P^{\frac{1}{2}}S(X_{i_0}^-)^{\frac{1}{2}}\ell\rangle + \\ \pi_{i_0}\left\langle \ell, X_{i_0}^-\ell\right\rangle + \pi_{i_0}^{-1}\langle \ell, Q\ell\rangle + \\ + \gamma\left(\sum_{j=1}^{m}\beta_{i_0 j}\left\langle \ell, X_j^-\ell\right\rangle - \left\langle \ell, X_{i_0}^-\ell\right\rangle\right) \geqslant \\ -2\|(X_{i_0}^-)^{\frac{1}{2}}\ell\|\,\|S\|\,\|P^{\frac{1}{2}}\ell\| + 2\left\langle \ell, X_{i_0}^-\ell\right\rangle^{\frac{1}{2}}\langle \ell, Q\ell\rangle^{\frac{1}{2}} = \\ = 2\left\langle \ell, X_{i_0}^-\ell\right\rangle^{\frac{1}{2}}(\rho(\ell \mid \mathscr{Q}) - \rho(\ell \mid \mathscr{P})).$$

Here we have used the Cauchy–Bunyakovsky–Schwarz inequality as well as relations $\|Sx\| \leq \|S\|\,\|x\|$, $2ab \leq a^2 + b^2$. The multiplier of γ is non-positive, since for $i = i_0$ the expression $\langle \ell, X_i^-\ell\rangle$ is at its maximum. Returning to the estimate of $\rho(\ell \mid \mathscr{X}^-[t-\sigma])$, we get relation

$$(\rho(\ell \mid \mathscr{X}^-[t+\sigma]) + \sigma\rho(\ell \mid \mathscr{Q}(t))) - \\ (\rho(\ell \mid \mathscr{X}^-[t]) - \sigma\rho(\ell \mid \mathscr{P}(t))) = o(\sigma),$$

which is equivalent to the funnel equation of the above. ■

Corollary 3: Set-valued function $\mathscr{X}^-[t]$ is an internal estimate of the reachability tube $\mathscr{X}[t]$, and functions $\mathscr{X}_i^-[t]$ are internal ellipsoidal estimates of $\mathscr{X}[t]$.

Proof: This follows from the fact that the reachability set is the maximum solution of the funnel equation with respect to inclusion. ■

Example 2: Figure 2 shows internal ellipsoidal estimates of the reachability set of an oscillating system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$ on time interval $[0, \pi]$ with $\gamma = \frac{1}{20}$ and $\frac{1}{2}$. Here exact approximations are degenerate ellipsoids (shown with thick lines).

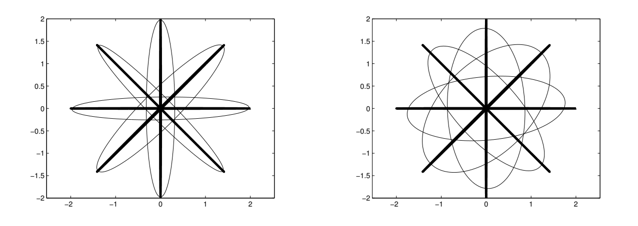

Fig. 2. Internal ellipsoidal estimates of the reachability set of a 2D system (left: $\gamma = \frac{1}{20}$, right: $\gamma = \frac{1}{2}$)

Example 3: Figure 3 depicts dependence of the size of ellipsoidal estimates for a higher-order oscillating system (number of nodes $N = 10$, dimension $n = 2N = 20$). On the left, 20 graphs of eigenvalues of the matrix $(X_1^-)^{\frac{1}{2}}$ are shown (these eigenvalues are semi-axes of the estimates). On the right the volume of the estimates is plotted in the power of $\frac{1}{n} = \frac{1}{20}$ (i.e. geometrical mean of the axes).

Analysing similar plots for a number of values of N, we came to conclusion that for robust computation of the reachability tube the number of approximations m should be chosen close to the dimension of the system n.

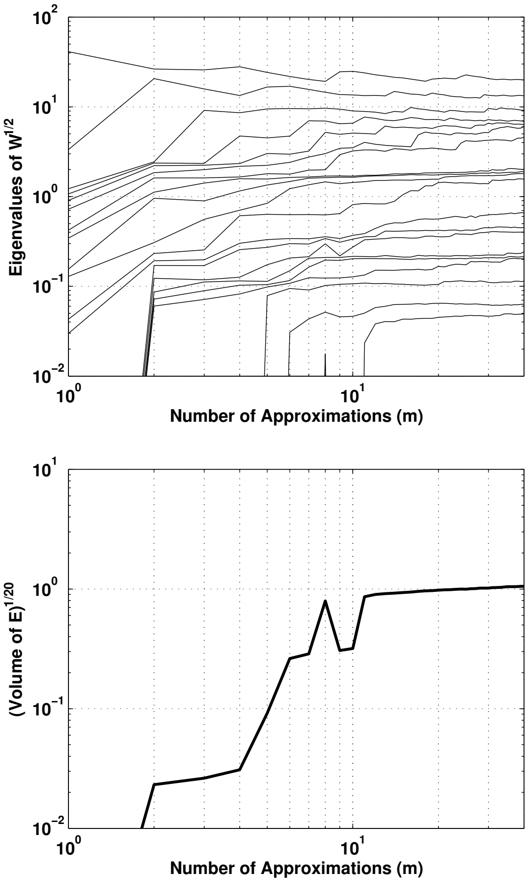

Fig. 3. The size of ellipsoidal estimates depending on the number of mixed estimates m (top: eigenvalues, bottom: volume in the power $\frac{1}{n}$)

III. Calculating Orthogonal Matrix S Efficiently

Equation (4) for the ellipsoidal approximation includes the operation of finding an orthogonal matrix $S = S(v_1, v_2) \in \mathbb{R}^{n\times n}$, such that $Sv_2 \upuparrows v_1$ for some non-zero vectors $v_1, v_2 \in \mathbb{R}^n$.

Note that with $n \geqslant 2$ the matrix $S(v_1, v_2)$ is not unique (for $n = 2$ there are at least two such matrices, and for $n \geqslant 3$ there are infinitely many).

Function $S(v_1, v_2)$ should be defined as sufficiently smooth in arguments v_1, v_2, so that higher-order ODE integration schemes could be applied to (4).

Matrix $S(v_1, v_2)$ may be calculated, for example, by computing the singular value decomposition of vectors v_1, v_2 and by multiplying the corresponding orthogonal matrices [6]. The number of operations for this procedure is of order $O(n^3)$, and continuous dependence of S on v_1, v_2 is not guaranteed.

The following theorem gives explicit formulas for calculating $S(v_1, v_2)$, using $O(n^2)$ operations, and sufficiently smooth in its arguments.

Theorem 5: Let $v_1, v_2 \in \mathbb{R}^n$ be some non-zero vectors. Then matrix $S \in \mathbb{R}^{n\times n}$ calculated as

$$S = I + Q_1(S - I)Q_1^T, \tag{9}$$

$$S = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \ c = \langle \hat{v}_1, \hat{v}_2 \rangle, \ s = \sqrt{1 - c^2}, \ \hat{v}_i = \frac{v_i}{\|v_i\|}, \tag{10}$$

$$Q_1 = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \in \mathbb{R}^{n\times 2}, \tag{11}$$

$$q_1 = \hat{v}_1, \quad q_2 = \begin{cases} s^{-1}(\hat{v}_2 - c\hat{v}_1), & s \neq 0, \\ 0, & s = 0, \end{cases}$$

is orthogonal and satisfies the property $Sv_2 \upuparrows v_1$.

Proof: Suppose $v_1 \nparallel v_2$ and pass to normalized vectors $\hat{v}_i = v_i / \|v_i\|$. We shall describe an orthogonal transformation S which is a rotation in plane π defined by vectors v_1 and v_2 that transforms vector $\hat{v}_2$ into $\hat{v}_1$. We impose an additional requirement that on orthogonal complement to π the induced operator $S|_{\pi^\perp}$ should be equal to identity.

Compose a matrix $V = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix}$ and find its QR decomposition: $V = QR$, where $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, and $R \in \mathbb{R}^{n\times 2}$ is an upper triangular matrix. Matrices Q and R may be written in block form as

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad Q_1 \in \mathbb{R}^{n\times 2}, \quad Q_2 \in \mathbb{R}^{n\times (n-2)},$$

$$R = \begin{bmatrix} R_1 \\ O_{(n-2)\times 2} \end{bmatrix}, \quad R_1 = \begin{pmatrix} 1 & c \\ 0 & s \end{pmatrix}.$$

Note that columns of Q_1 and Q_2 form an orthonormal basis on the plane π and its orthogonal complement $\pi^\perp$ respectively. With an additional constraint $R_{11} > 0$, $R_{22} > 0$ the matrix Q_1 is unique, and the matrix Q_2 may be arbitrary with orthonormal columns orthogonal to those of Q_1. Relations (11) may be regarded as the Gram–Schmidt orthogonalization process for finding Q_1.

Set

$$S = Q \begin{bmatrix} S & O \\ O & I \end{bmatrix} Q^T.$$

This matrix is orthogonal as the product of orthogonal matrices. It describes the composition of three transformations: a rotation of plane π to plane $\mathscr{L}(e_1, e_2)$, a rotation in that plane, and a return to original coordinates.

We now prove that S does not depend on Q_2. Indeed, $S = Q_1 S Q_1^T + Q_2 Q_2^T$. But from equality $QQ^T = Q_1 Q_1^T + Q_2 Q_2^T = I$ it follows that $Q_2 Q_2^T = I - Q_1 Q_1^T$, so we get (9), which does not contain Q_2.

Matrix S is also orthogonal when $v_1 \parallel v_2$, i.e. $s = 0$. In this case $c = \pm 1$. For $c = 1$ we have $S = I$, hence $S = I$. For $c = -1$ we get $S = I - 2\hat{v}_1\hat{v}_1^T$, and check that $SS^T = I$.

Now calculate $S\hat{v}_2$. From (11) we get $Q_1^T \hat{v}_2 = \begin{pmatrix} c \\ s \end{pmatrix}$ for any value of s. We further have

$$S\hat{v}_2 = \hat{v}_2 + Q_1 \begin{pmatrix} c-1 & s \\ -s & c-1 \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} = \hat{v}_2 + \begin{bmatrix} \hat{v}_1 & s^{-1}(\hat{v}_2 - c\hat{v}_1) \end{bmatrix} \begin{pmatrix} 1-c \\ -s \end{pmatrix} = \hat{v}_1.$$

For any vector $v \perp \hat{v}_1, \hat{v}_2$ we have $Q_1^T v = 0$, hence $Sv = v$. ■

Remark 5: The number of operations in (9)–(11) is of order $O(n^2)$. Moreover, multiplication by matrix S may be performed with $O(n^2)$ operations instead of usual $O(n^3)$.

Remark 6: It is straightforward to check that mapping $S = S(v_1, v_2)$ has continuous derivatives of any order everywhere except the cone $v_1 \upharpoonleft\downharpoonright v_2$.

IV. Parallel Computations

Here we discuss issues of parallel computations for finding ellipsoidal estimates and solving the feedback control problem.

A. Calculating the Ellipsoidal Estimates

In order to solve the ODE (8) numerically, on μ parallel processes, we decompose the index set $I = \{1, \dots, m\}$ into μ disjoint subsets I_k, such that $I = I_1 \cup \dots \cup I_\mu$. Process k will calculate and store matrices X_i^-, $i \in I_k$. (For load balancing purposes the number of elements in subsets I_k should be approximately the same, close to m/μ.)

Integrating ODE (8) "as it is" would lead to an excessive amount of data exchange between processes in order to compute the term $\sum_{i=1}^m \beta_{ij} X_j^-$. To avoid this, we combine (8) with (6). Each process solves its own ODE (8) with the mentioned term replaces with $\sum_{i \in I_k} \beta_{ij} X_j^-$, where $\sum_{j \in I_k} \beta_{ij} = 1$. Estimates belonging to different processes are mixed in discrete time points using (6) with $\alpha_i = \frac{1}{m}$.

In other words, ellipsoidal approximations are mixed by (8) in each process separately, and all approximations are mixed by (6) at discrete time instants.

B. Calculating the Feedback Control

For the feedback control problem the exact "extremal aiming" control should be calculated as

$$\mathscr{U}^*(t,x) = \begin{cases} -\frac{P(t)B^T(t)p}{\langle B^T(t)p, P(t)B^T(t)p\rangle^{\frac{1}{2}}}, & B^T(t)p \neq 0; \\ \mathscr{P}(t), & B^T(t)p = 0; \end{cases}$$

where the vector $p = p(t,x) = \partial V^-/\partial x$, $V^-(t,x) = d(G(t_1,t)x, G(t_1,t)\mathscr{W}^-[t])$ is the direction of the shortest path from x to $\mathscr{W}^-[t]$. The set $\mathscr{W}^-[t]$ is the convex hull of a union of sets, therefore the calculation of vector p reduces to a computationally difficult max-min problem

$$V^-(t,x) = \max_{\|p\|\le 1} \min_{i=1,\dots,m} \Big\{ \langle p, G(t_1,t)x\rangle - \rho\big(G^T(t_1,t)p \,\big|\, \mathscr{W}_i^-[t]\big)\Big\} \quad (12)$$

(since parameters of the sets are scattered across multiple processes).

To overcome this difficulty, we replace $V^-(t,x)$ with

$$\hat{V}^-(t,x) = d(G(t_1,t)x, G(t_1,t)\hat{\mathscr{W}}^-[t]),$$
$$\hat{\mathscr{W}}^-[t] = \bigcup_{i=1}^m \mathscr{W}_i^-[t] = \bigcup_{i=1}^m \mathscr{E}(w(t), W_i(t)),$$

which is equivalent to interchanging the maximum and minimum in (12)

$$\hat{V}^-(t,x) = \min_{i=1,\dots,m} \max_{\|p\|\le 1} \Big\{ \langle p, G(t_1,t)x\rangle - \rho\big(G^T(t_1,t)p \,\big|\, \mathscr{W}_i^-[t]\big)\Big\}. \quad (13)$$

This leads to a vector $\hat{p}(t,x) = \partial V^-/\partial x$ equal to $\hat{p} = \hat{p}_{i_0}$, where $i_0 \in \overline{1,m}$ is the minimizer index in (13), and $\hat{p}_i$ is the maximizer in (13) for that fixed $i = i_0$.

This way, each process locally finds an ellipsoid nearest to the current state. Afterwards the process that holds the nearest possible ellipsoid calculates the value of control by using that ellipsoid, and communicates the control to all the other processes.

C. Results of Modeling

Our software implementation is written in Fortran 2003 using MPI (Message Passing Interface) for parallelization, MKL (Intel Math Kernel Library) for matrix operations, and NAG library for solving ODEs and other types of calculations.

In a series of numerical experiments, we were able to solve feedback control problems for an oscillating system [5] with the following parameters (N is the number of oscillators, dimension of the system is $n = 2N$):

- $N = 25$ ($n = 50$) for a system with disturbance, without matching condition;
- $N = 50$ ($n = 100$) for a system with unilateral scalar control ($u \in [0, \mu]$);
- $N = 50$ ($n = 100$) for a non-homogeneous oscillating system;
- $N = 100$ ($n = 200$) for a system with bilateral scalar control;
- $N = 250$ ($n = 500$) for a system with vector control of dimension N.

We do not compare these results to a non-parallel version of our code, since existing memory limitations prevent it from being run with large values of n.

In our previous experiments without mixing of ellipsoidal estimates [5] we were able to achieve $n = 25$ for a system with matching condition. There we were unable to further increase n due to the degeneracy of estimates (which was now addressed by the methods of this paper).

V. Conclusion

This paper presents an innovative scheme for calculating the reachability set under uncertainty with applications to the problem of feedback control. Examples of its efficiency are demonstrated on systems of dimension up to 500.

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