Estimation of Reachability Sets for Large-Scale Uncertain Systems: from Theory to Computation

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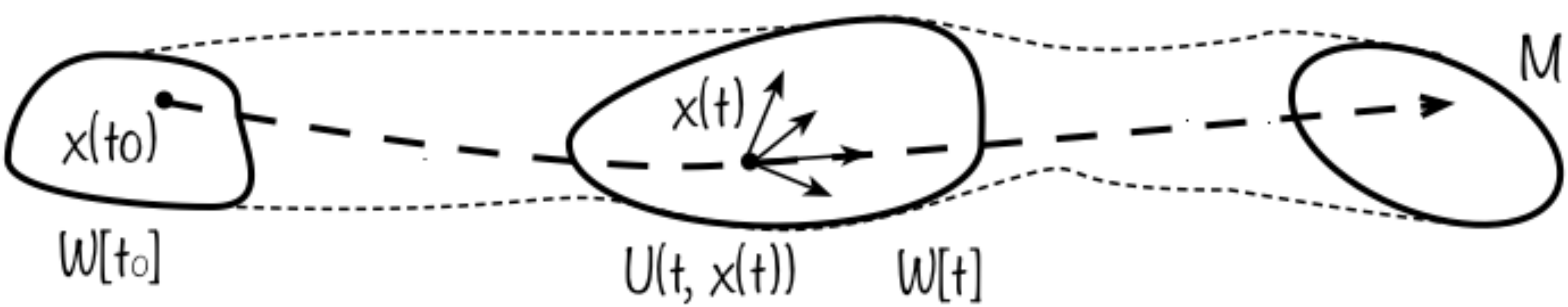
Overview of the Problem

- Find **Reachability Set** of a Linear System
  - High Dimension (hundreds of variables)
  - Uncertainty
- Approach: **Ellipsoidal Calculus**
  - Has issues if applied «as is»
- New technique: **Mix Ellipsoidal Estimates**
  - Stable calculation of reach sets for large systems
  - Parallel computations

Mathematical Formulation

- System: $\dot{x} = A(t)x + u + v, \quad t \in [t_0, t_1]$
- Constraints: $u(t) \in \mathcal{P}(t), \quad v(t) \in \mathcal{Q}(t)$
- Objective: $x(t_1) \in \mathcal{M}$
- Feedback control: $\mathcal{U}(t, x)$
- Backward reachability set: $\mathcal{W}[t]$



Ellipsoidal Estimates

- Ellipsoid: $\mathcal{E}(q, Q)$

$$\rho(\ell \mid \mathcal{E}(q, Q)) = \langle \ell, q \rangle + \langle \ell, Q\ell \rangle^{\frac{1}{2}}$$

$$\mathcal{E}(q, Q) = \{x \mid \langle x - q, Q^{-1}(x - q) \rangle \leq 1\}$$

- Ellipsoidal Constraints:

$$\mathcal{P}(t) = \mathcal{E}(0, P(t)), \quad \mathcal{Q}(t) = \mathcal{E}(0, Q(t)), \quad \mathcal{M} = \mathcal{E}(0, M)$$

Ellipsoidal Estimates

$$\mathcal{W}^-[t] = \mathcal{E}(0, W(t))$$

- ODEs for **Inner Ellipsoidal Approximation**

A. B. Kurzhanski, P. Varaiya // Optim. Meth. & Software, V. 17, N. 2, 2002.

A. A. Kurzhanski, P. Varaiya. Ellipsoidal Toolbox, 2005.

$$\dot{W} = AW + WA^T - W^{\frac{1}{2}} T P^{\frac{1}{2}} - P^{\frac{1}{2}} T^T W^{\frac{1}{2}} + \pi W + \pi^{-1} Q$$

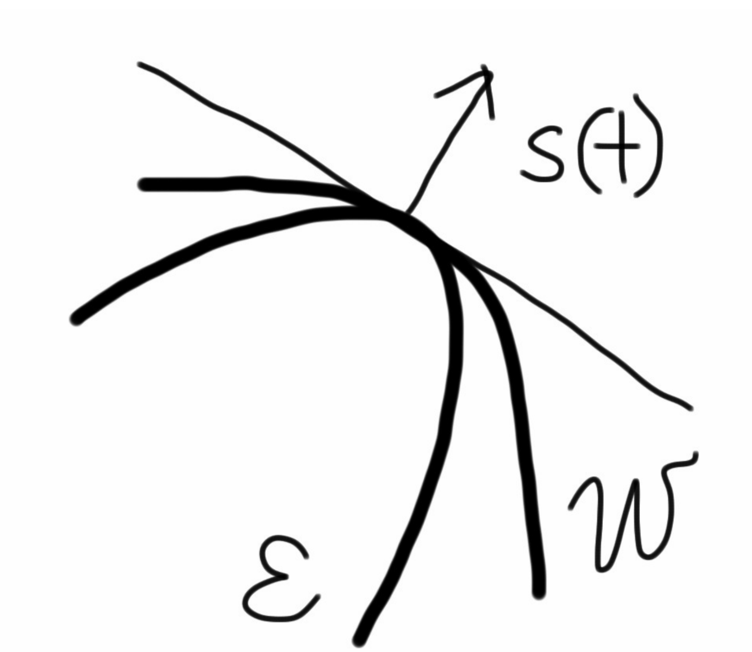
$$TT^T = I, \quad TP^{\frac{1}{2}} s \parallel W^{\frac{1}{2}} s \qquad \dot{s} = -A^T s, \quad s(t_1) = \ell$$

$$\pi = \frac{\langle s, Qs \rangle^{\frac{1}{2}}}{\langle s, Ws \rangle^{\frac{1}{2}}} \qquad W(t_1) = M$$

Ellipsoidal Estimates

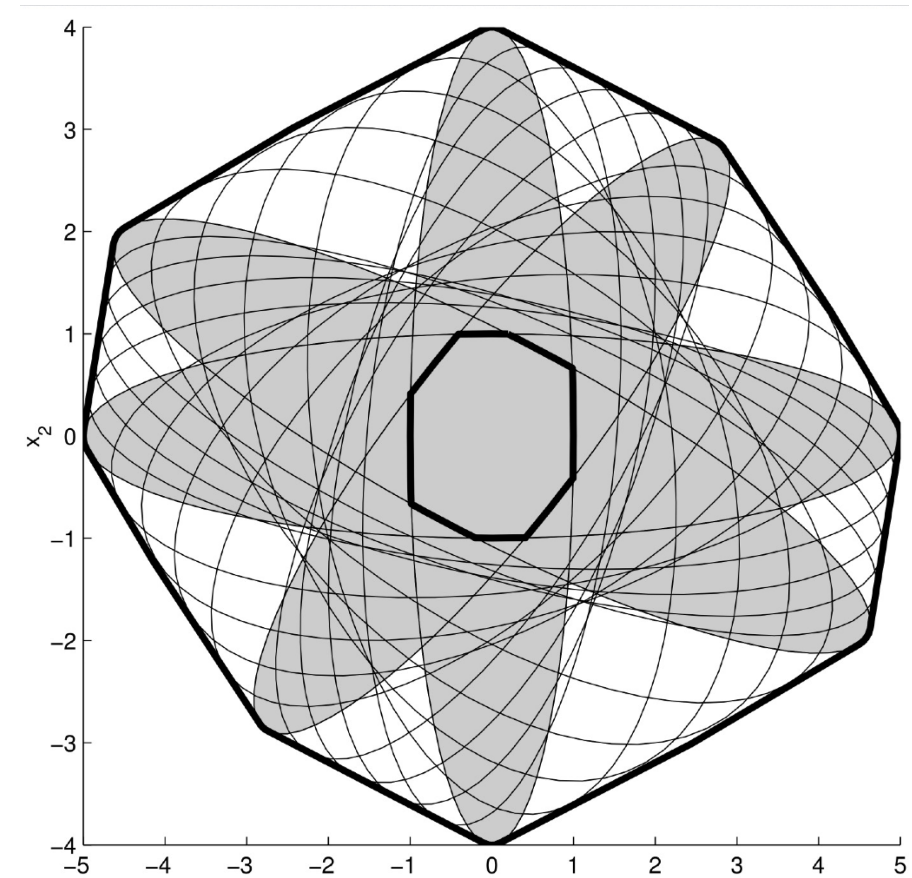
- Estimates are **Tight**:

$$\rho(s(t) \mid \mathcal{W}^{-}[t]) = \rho(s(t) \mid \mathcal{W}[t])$$



- **Family** of Estimates

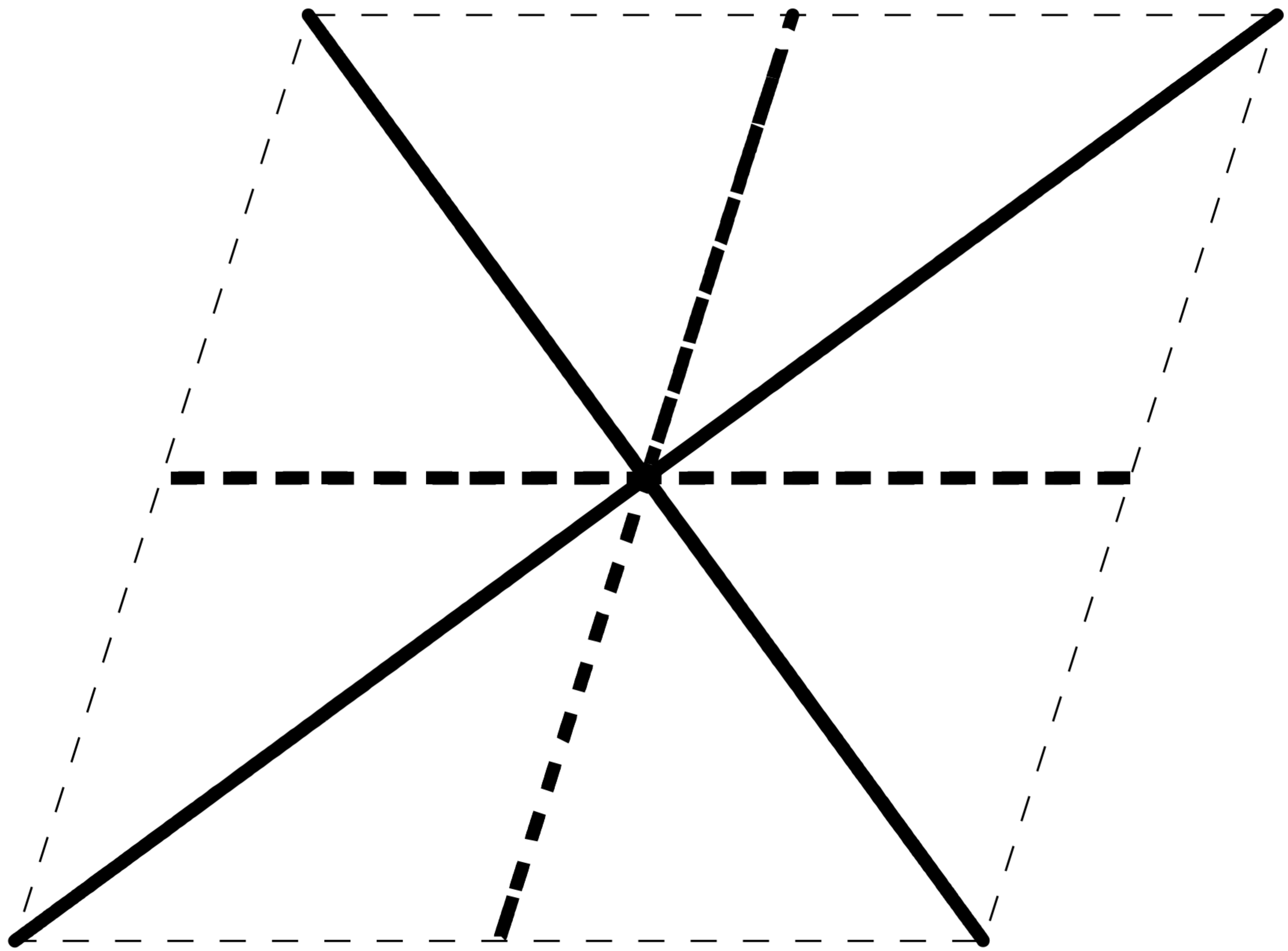
$$\ell = \ell_i, \quad i = 1, \dots, k$$



Large-Scale Systems

- Ill-Conditioned Estimates
  - Need Regularization => **Mixing the Estimates**
- Need for **Parallel Computations**
  - P.C. is natural for ellipsoidal methods

Degeneracy of Estimates



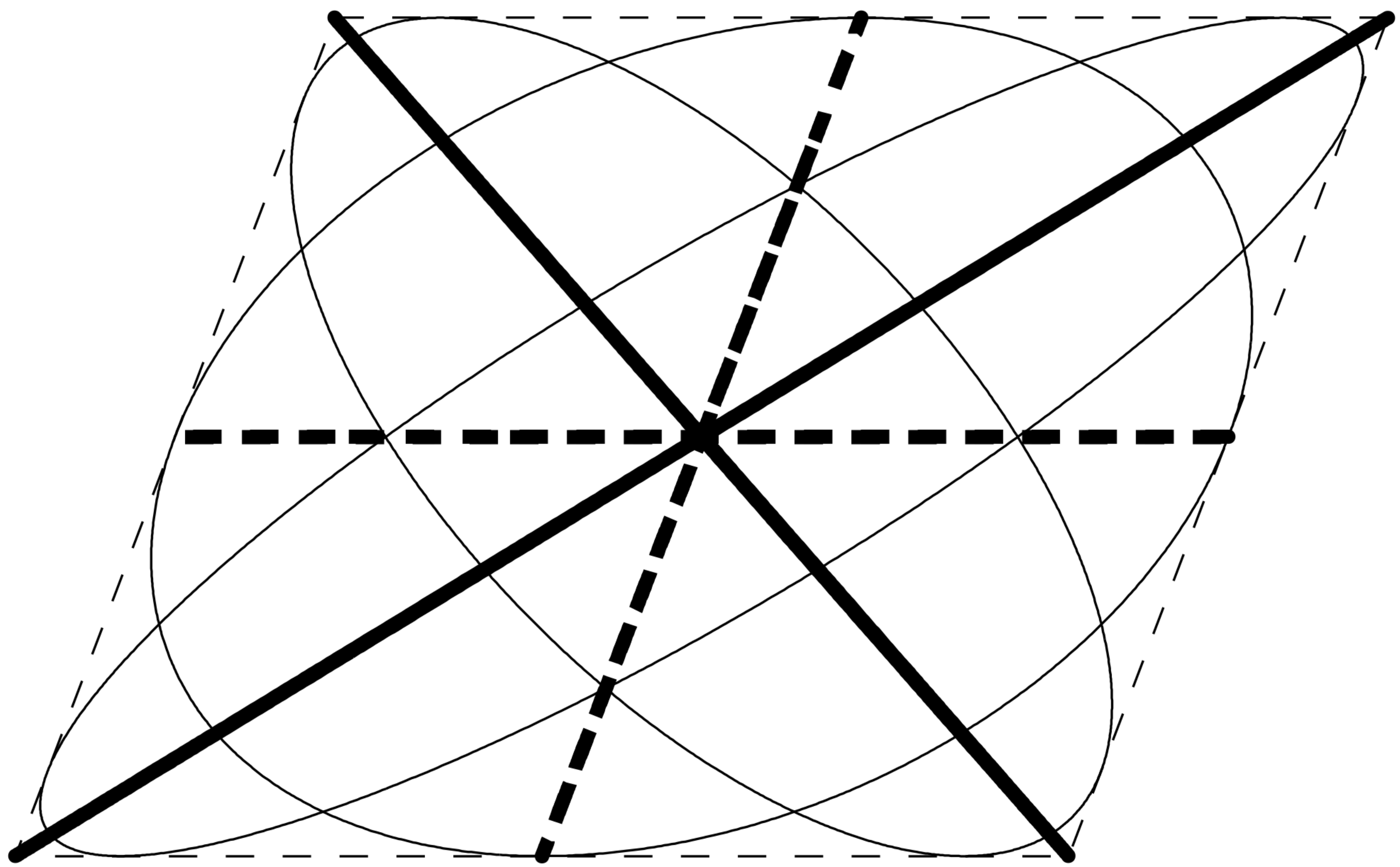
Ideas

- In this case only degenerate estimates are tight
    - **Relax tightness.**
      (but don't drop this requirement!)

- Estimates form a parameterized family
    - **Combine estimates**

Mixing: Sum of Ellipsoids

$$Q = \alpha Q_1 + (1 - \alpha) Q_2$$



Mixing: Reachability Set

$$W_i(t_1) = M, \quad i = 1, \ldots, m$$

$$\dot{W}_i = AW_i + W_i A^T - W_i^{\frac{1}{2}} T_i P^{\frac{1}{2}} - P^{\frac{1}{2}} T_i^T W_i^{\frac{1}{2}} + \pi_i W_i + \pi_i^{-1} Q + \\ + \gamma \left(\sum_{j=1}^m \beta_{i,j} W_j(t) - W_i(t) \right)$$

$$T_i T_i^T = I, \quad T_i P^{\frac{1}{2}} s_i \parallel W_i^{\frac{1}{2}} s_i$$

$$\dot{s}_i = -A^T s_i, \quad s_i(t_1) = \ell_i$$

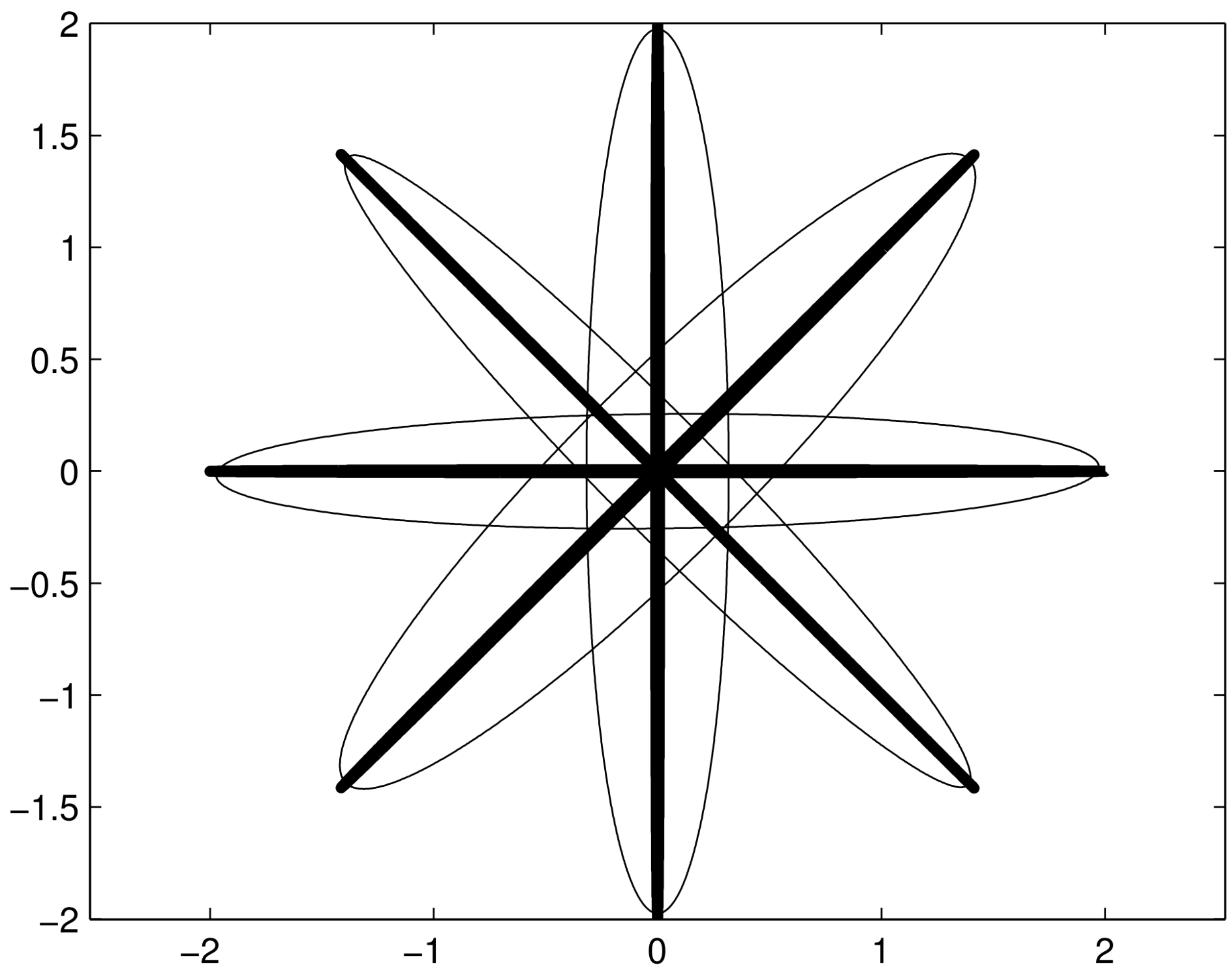
$$\pi_i = \frac{\langle s_i, Q s_i \rangle^{\frac{1}{2}}}{\langle s_i, W_i s_i \rangle^{\frac{1}{2}}}$$

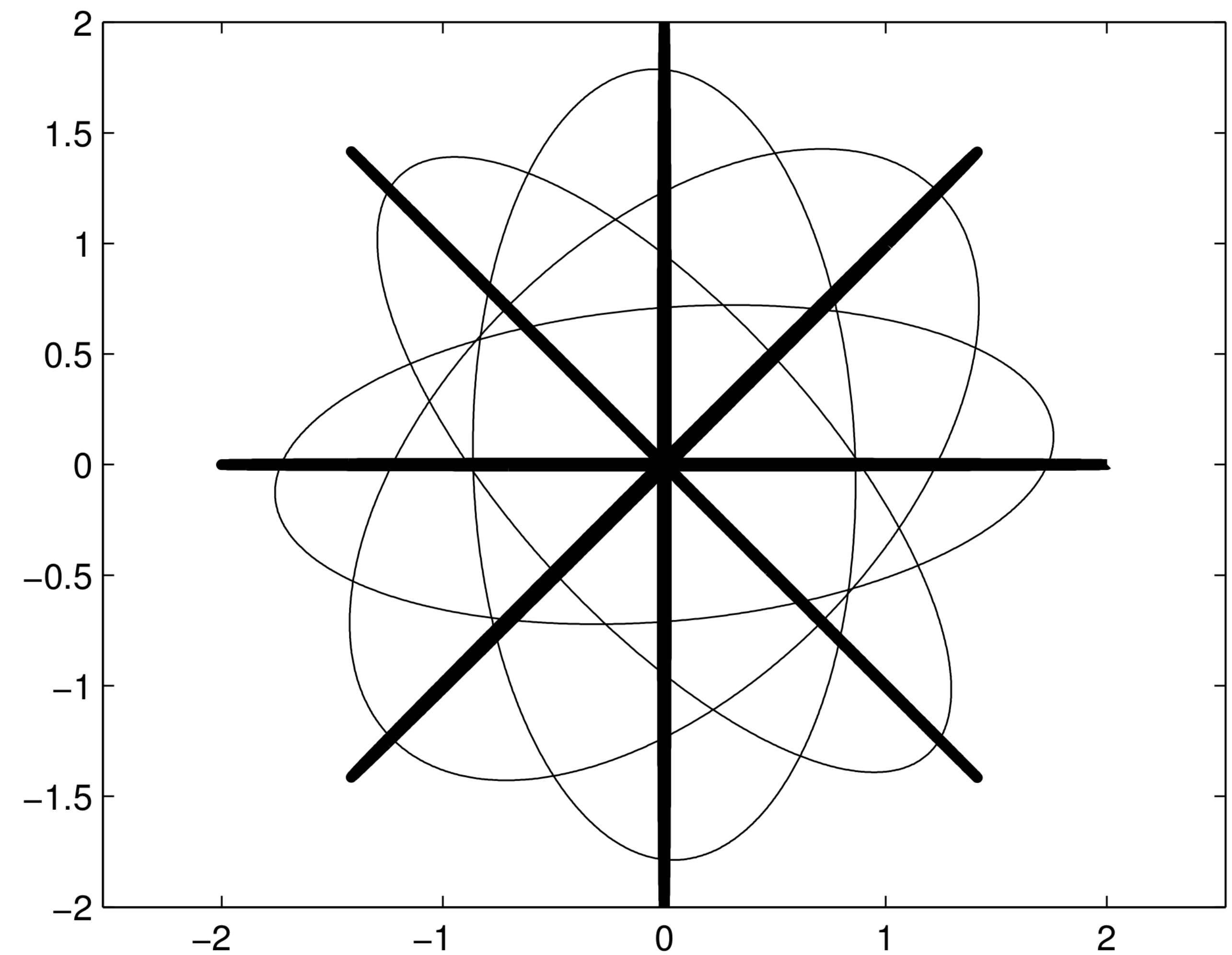
$$W_i(t_1) = M$$

$$\sum_{j=1}^m \beta_{ij} = 1$$

$$\gamma, \beta_{i,j} \geq 0$$

Examples





Properties of Mixed Estimates

- $\mathcal{W}_i^-[t] \subseteq \mathcal{W}[t]$
- Funnel Equation: $\mathcal{W}^-[t] = \operatorname{conv} \bigcup_{i=1}^m \mathcal{W}_i^-[t]$

$$\lim_{\sigma \to 0+} \sigma^{-1} h_+((I + \sigma A)\mathcal{W}^-[t - \sigma] + \sigma \mathcal{Q}, \mathcal{W}^-[t] - \sigma \mathcal{P}) = 0$$

- HJBI Inequality: $V_-(t, x) = d(x, \mathcal{W}^-[t])$

$$\min_{u \in \mathcal{P}} \max_{v \in \mathcal{Q}} \{V_t^- + \langle V_x^-, Ax + u + v \rangle\} \leq 0$$

Parallel Computations

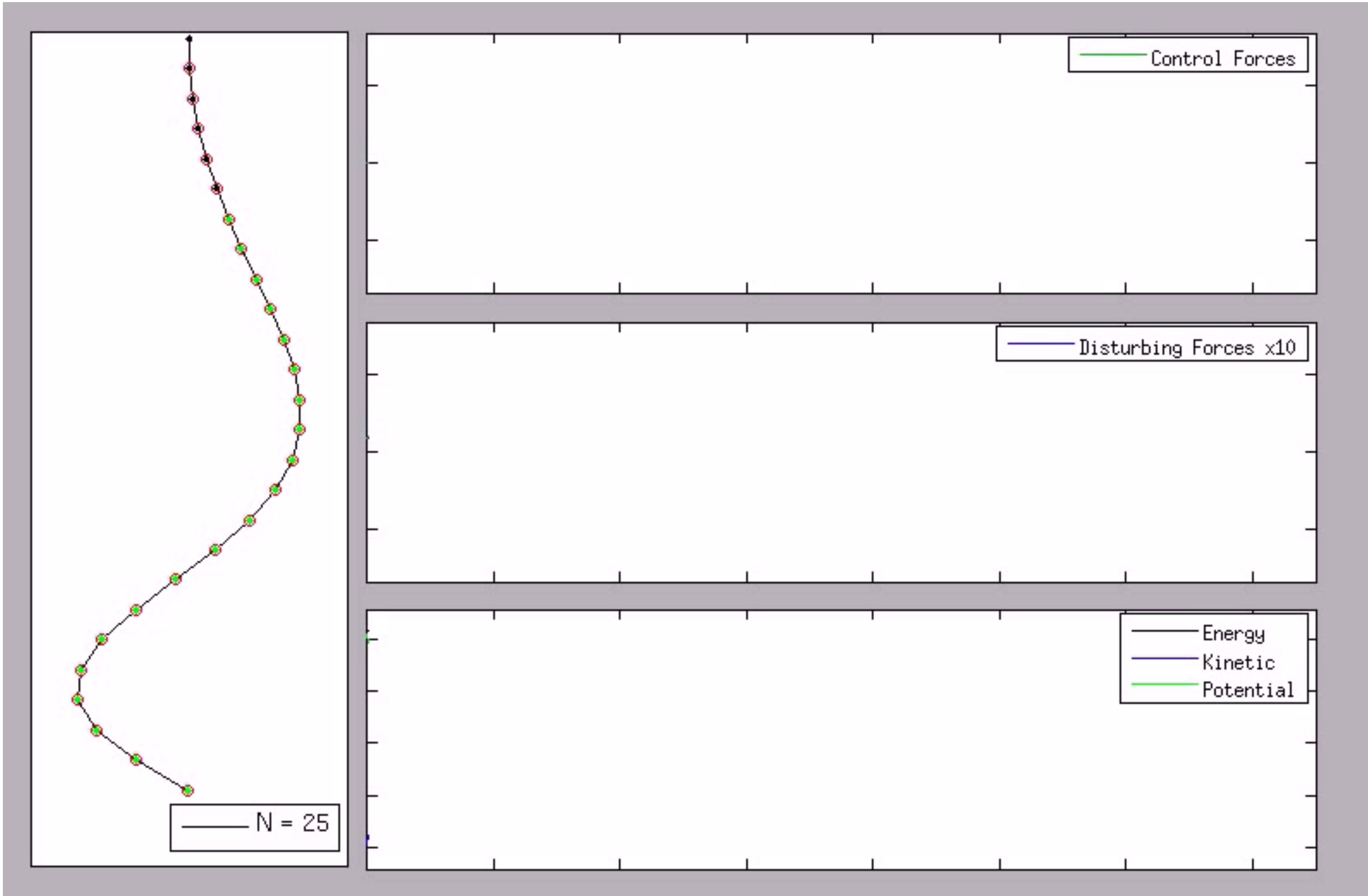
- Estimates are divided between processes
- Within each process, mix estimates using proposed ODE
- After some time mix estimates between processes

Numerical Results

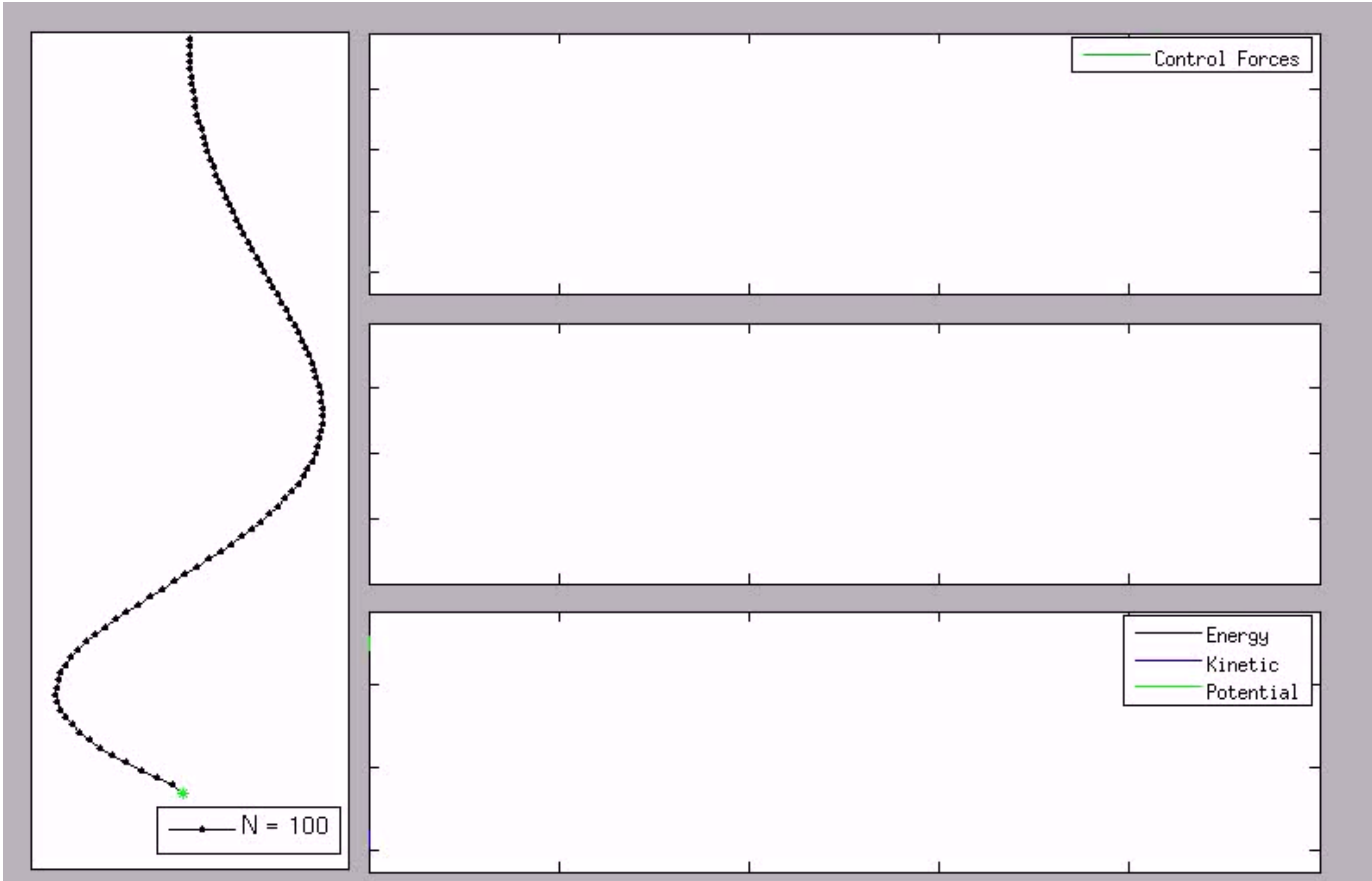
- For a large-scale oscillating system ***n*=2*N***, N is the number of links:

- ***N*=25** under uncertainty, without matching condition
- ***N*=50** for a heterogeneous system
- ***N*=50** in case of unilateral control
- ***N*=100** for scalar control
- ***N*=250** for vector control of dimension N

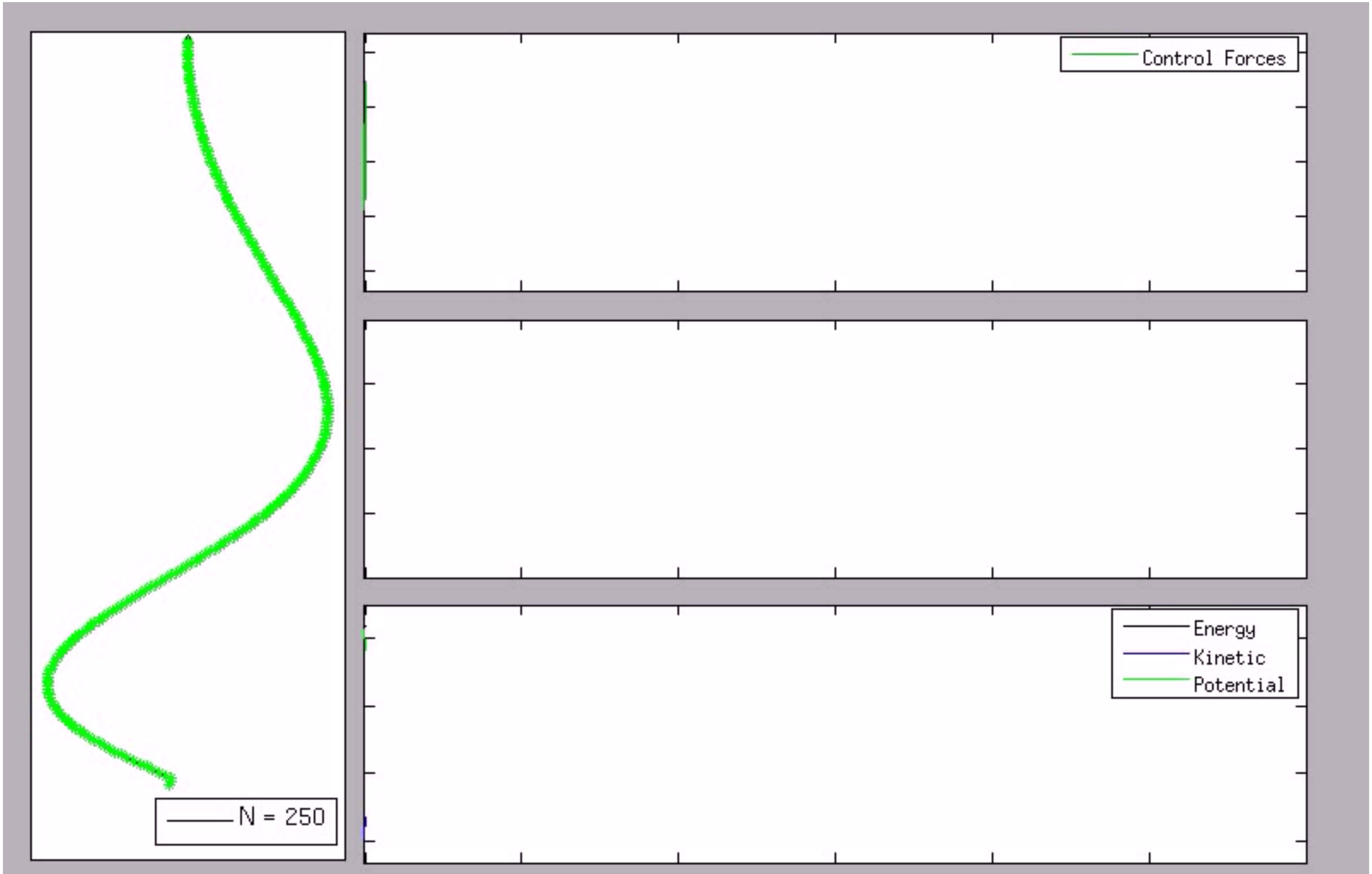
Examples: N=25, Vector Disturbance



Examples: N=100, Scalar Control



Examples: N=250 (n=500)



Thank you!