Output Feedback Strategies for Systems with Impulsive and Fast Controls

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***Abstract*— This paper deals with *output feedback impulse control* under set-membership uncertainty where the control realization consists of a sequence of δ-impulses. It indicates solution schemes based on generalized Dynamic Programming relations of the HJB type and suggests recommendation for computation. The problem is then generalized to the case of high-order "fast" controls which solve the terminal control problem in arbitrary small time. Finally an output feedback control problem is solved where communication signals for the available noisy measurements arrive at Poisson instants of time. Numerical examples are demonstrated.**

I. Introduction

The problem of output feedback control through available measurements under uncertain disturbances (noise) is one of the central issues of control theory. It was thoroughly developed within a stochastic model with statistical information on the noise [1], [2]. However a large array of pending problems are to be solved with no such information, but only under a set-membership description of the uncertain items [3]–[7]. This paper deals with one of such problems whose specifics are also in the fact that *the output feedback is to be generated by impulse controls*. The theory of closed-loop impulse control, initiated in [8], was developed in papers [9], [10] where considered were not only impulsive inputs of δ-type but also impulses of higher order, described by higher derivatives of δ-functions. Such inputs describe virtual controls which can solve the terminal control problem in finite time equal to zero. Their physically realizable approximations allow to solve such problems in arbitrary small "nano-time".

Alternatively considered are problems of closed-loop impulse control with measurement signals arriving only at random instants of time while satisfying a Poisson distribution and being corrupted by bounded stochastic noise. Here stochasticity is intertwined with a set-membership approach.

II. The Problem of Output Feedback Impulse Control

A. The Initial Formulation

In this section we start by an impulse control problem modeled by one of minimizing a generalized Meier–Bolza-type functional:

$$\begin{cases} J(u(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1+0)) \to \inf, \\ dx(t) = A(t)x(t)dt + B(t)dU(t), \quad t \in [t_0,t_1]. \end{cases} \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, $U(\cdot) \in BV\,([t_0,t_1];\mathbb{R}^m)$ is the generalized control, $BV\,([t_0,t_1];\mathbb{R}^m)$ is the space of m-vector functions of bounded variation (here we assume that all functions of bounded variation are left-continuous). Matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ are assumed continuous. The terminal time t_1 is fixed. The terminal function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed and convex.

The information available to control is given by the following measurement equation:

$$y(t) = H(t)x(t) + \xi(t). \tag{2}$$

Here $\xi(t) \in \mathscr{Q}(t)$ is the bounded noise, $\mathscr{Q}(t) \in \operatorname{conv}\mathbb{R}^n$.

For this system we are to solve a particular case of the measurement feedback control problem [4]. The particular properties of this simplified version allow to present a fairly simpler solution scheme than in the more general case. Here is a preliminary loose version of the problem.

Problem 1: For a given terminal function $\varphi(\cdot)$, find a measurement impulse feedback control minimizing the functional $J(U(\cdot))$, despite the disturbance $\xi(\cdot)$, i.e. minimizing

$$\mathscr{J}(U(\cdot)) = \max\{J(U(\cdot)) \mid x(\cdot)\},$$

where the maximum is taken over all trajectories consistent with available measurements (2).

B. The Controlled System

In our initial formulation, the measurement $y(t)$ is a discontinuous function even if the noise is smooth, due to impulses in control. Here we perform a decomposition of the initial system into two parts, the one with uncertainty and the other with impulses.

Let $X(t,\tau)$ be the transition matrix of the homogeneous system, a solution to the matrix equation $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.

We represent the variable $x(t)$ as a sum

$$x(t) = X(t,t_1)x_1(t) + X(t,t_1)x_2(t), \tag{3}$$

where $x_1(t)$ and $x_2(t)$ satisfy the system

$$\begin{cases} \dot{x}_1(t) = 0, \\ dx_2(t) = B(t)dU(t), \quad x_2(t_0) = 0, \end{cases} \tag{4}$$

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and the measurement equation becomes

$$y_1(t) = H(t)X(t,t_1)x_1(t) + \xi(t) = \\ = y(t) - H(t)X(t,t_1)x_2(t).$$

We see that the equation for $x_2(t)$ does not contain any uncertain items, hence $x_2(t)$ is known to the controller. The measurement $y_1(t)$ is also expressed through available information — the actual measurement $y(t)$ and the known variable $x_2(t)$.

C. The Information State and the Information Set

Now we need to define *the state* of the system with feedback noise. According to [4] here we distinguish two problems — the one of *guaranteed state estimation* which gives us this state and the problem of *feedback control in the space of states*.

Suppose the measurement process begins at $t_0 < t_1$. Then the on-line *information state* or *position* of the system is defined as the triplet $\{t, \mathscr{X}_1[t], x_2(t)\}$ where $\mathscr{X}_1[t]$ is the *information set* of all possible states $x_1(t)$ consistent with the system model, the available measurement $y_1(s)$, $s \in [t_0, t]$, $t \le t_1$, and the constraint $\mathscr{Q}$ on the unknown but bounded noise $\xi(\cdot)$.

Set $\mathscr{X}_1[t]$ is the solution to the problem of guaranteed state estimation [5], [6]. Here

$$\mathscr{X}_1[t] = \bigcap \{ H^{-1}(t)\left(y_1^*(\tau) - \mathscr{Q}(\tau)\right) \mid \tau \in [t_0, t]\},$$

where realizations of measurements $y_1^*(t)$ on time interval $t \in [t_0, t]$ are given.

The information set may be also described through its support function or funnel equation.

The support function[1] of the set $\mathscr{X}_1[t]$ may be calculated through techniques of convex analysis (see [4], [6], [11]). Note that the measurement equation produces at $t = t_0$ the inclusion $x_1(t_0) \in H^{-1}(t_0)(y_1^*(t_0) - \mathscr{Q}(t_0)) = \mathscr{X}_1^0$. We have

$$\rho(\ell \mid \mathscr{X}_1[t]) = \inf_{\lambda(\cdot)} \left\{ \int_{t_0}^{t} \big(\langle \lambda(\tau), y_1^*(\tau)\rangle + \\ + \rho(-\lambda(\tau) \mid \mathscr{Q}(\tau))\big) d\tau \mid \psi(t) = \ell \right\},$$

where the vector row ψ satisfies the differential equation

$$\dot{\psi} = \lambda(t)H(t), \qquad \psi(t_0) = 0.$$

If the disturbance and hence the measurements are smooth enough, then $\mathscr{X}_1[t]$ is the solution to the *funnel equation*[2] [12]:

$$\lim_{\sigma \to 0+0} \sigma^{-1} h\big(\mathscr{X}_1[t+\sigma], \\ \mathscr{X}_1[t] \cap H^{-1}(t)(y_1^*(t) - \mathscr{Q}(t))\big) = 0. \quad (5)$$

[1]The support function of a convex set A is

$$\rho(\ell \mid A) = \max\{\langle \ell, x\rangle \mid x \in A\}.$$

[2]Here $h(A,B)$ is the Hausdorff distance between two compacts: $h(A,B) = \max\{h_+(A,B), h_-(A,B)\}$, $h_+(A,B) = \min\{\varepsilon \mid A \subseteq b + \varepsilon \mathscr{B}_1\}$, $h_-(A,B) = h_+(B,A)$.

The information state thus consists of the information set $\mathscr{X}_1[t]$ which does not depend on the control and the vector $x_2(t)$ which has to be controlled in such a way that the sum $x_2(t) + \mathscr{X}_1[t]$ would be steered towards the terminal set $\mathscr{M}$ with uncontrollable component $\mathscr{X}_1[t]$ being estimated on-line.

D. The Precise Formulation of the Problem

Thus we have the set-valued information state $\{t, \mathscr{X}_1, x_2\}$, where $\mathscr{X}_1$ is convex and compact. To proceed further with the *the measurement feedback control problem* while trying to mimic traditional theory under complete information we would have to deal with control problems in the *metric space* of convex compact sets, [4], [7]. But the problem treated here allows a fairly simpler solution and may be reduced to one in finite-dimensional space. We may now give a more precise formulation for Problem 1.

Problem 2: Given position $\{t, \mathscr{X}_1, x_2\}$, $t \in [t_0, t_1]$, indicate an impulse feedback control strategy which minimizes the cost

$$\mathscr{J}(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi\left(\mathscr{X}_1[t_1] + x_2(t_1+0)\right), \\ \varphi(\mathscr{X}) = \max\{\varphi(x) \mid x \in \mathscr{X}\},$$

whatever be the measurement $y_1(t)$ (that is, despite the unknown measurement noise $\xi(t) \in \mathscr{Q}(t)$).

E. The Solution

Let $V(t, x; t_1, \varphi(\cdot))$ be the value function in the impulse control problem (see [9], [13]) with given terminal functional $\varphi(x)$:

$$V(t,x;t_1,\varphi(\cdot)) = \min_{U(\cdot)} \Big\{ \operatorname*{Var}_{[t,t_1]} U(\cdot) + \varphi(x(t_1+0)) \mid \\ x(t) = x,\ dx(\tau) = B(\tau) dU(\tau) \Big\}. \quad (6)$$

Introduce a linear mapping $\mathbf{T}U[\tau, t] = x_2(t+0) - x_2(\tau)$ (for $\tau = t_0$ we simply have $\mathbf{T}[t_0, t] = x_2(t{+}0)$). We estimate the minimum terminal cost $\mathscr{V}(t, \mathscr{X}_1, x_2)$ for Problem 2 as

$$\mathscr{V}(t, \mathscr{X}_1, x_2) \le \min\{\mathscr{J}(U(\cdot)) \mid U(\cdot)\} \le \\ \le \min_{U(\cdot)} \left\{ \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(\mathscr{X}_1 + x_2 + \mathbf{T}U[t, t_1+0]) \right\} = \\ = V(t, 0; t_1, \boldsymbol{\varphi}(\cdot)),$$

where $\boldsymbol{\varphi}(x) = \max\{\varphi(x+z) \mid z \in \mathscr{X}_1 + x_2\}$.

In particular, if $\varphi(x) = I(x \mid \mathscr{M})$, then[3]

$$\boldsymbol{\varphi}(x) = I(x \mid \mathscr{M} \dot{-} (\mathscr{X}_1 + x_2)).$$

Here $\dot{-}$ denotes the geometric (Minkowski) difference between two convex sets: $A \dot{-} B = \{x \mid B + x \subseteq A\}$.

The value function $V(t, x; t_1, \varphi(\cdot))$ is the solution to the following variational inequality of Hamilton–Jacobi–Bellman type [9]:

$$\min\{H_1(t, x, V_t, V_x),\ H_2(t, x, V_t, V_x)\} = 0, \quad (7)$$

[3]By $I(x \mid A)$ we denote the indicator function of the convex set A (zero in A and $+\infty$ outside of A).

with initial condition $V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot))$ and the Hamiltonians

$$H_1(t, x, \xi_t, \xi_x) = \xi_t,$$
$$H_2(t, x, \xi_t, \xi_x) = \min\{\langle \xi_x, B(t)u\rangle + 1 \mid u \in S(0)\}.$$

Here $S(0)$ is the unit sphere in $\mathbb{R}^n$.

Due to (7), in any position (t, x) there are two possibilities for the control. Either $H_1 = 0$, and the control may choose $dU(t) = 0$, or $H_1 > 0$, in which case it is necessary that $H_2 = 0$, and the control has a jump in direction $-B'(t)V_x$. The magnitude of the jump is to be selected in such a way that after the jump we again have $H_1 = 0$.

The computation of the value function $V(t, x)$ relies on the following representation due to convex analysis:

$$V(t,x) = \sup[\langle p, x\rangle - \varphi^*(p) - I(p \mid \mathscr{B}_{\|\cdot\|_{[t,t_1]}}) \mid p \in \mathbb{R}^n] = \\ = \sup[\langle p, x\rangle - \varphi^*(p) \mid p \in \mathscr{B}_{\|\cdot\|_{[t,t_1]}}]. \quad (8)$$

Here φ^* denotes the Fenchel conjugate of φ, [11], and $\mathscr{B}_{\|\cdot\|_{[[t,t_1]]}}$ is the unit ball in the vector norm

$$\|p\|_{[t,t_1]} = \left\|B^T(\cdot)p\right\|_{C[t,t_1]} = \max_{\tau\in[t,t_1]} \left\|B^T(\tau)p\right\|. \quad (9)$$

It is straightforward to check that the function (8) indeed satisfies the variational inequality (7).

In order to evaluate (8) numerically, one replaces the maximum over $[t, t_1]$ in (9) with a maximum over a finite number of time instants, and the condition $\left\|B^T(\tau)p\right\| \le 1$ (for each of these instants) with a finite number of linear inequalities of type $\left\langle \ell_i, B^T(\tau)p\right\rangle \le 1$ (with vectors ℓ_i, $i = \overline{1, N}$, from the unit sphere) which gives a finite-dimensional optimization problem with a finite number of linear constraints (see [9] for details).

We summarize the above results in the following theorem.

Theorem 1: The optimal value of the functional $\mathscr{J}(U(\cdot))$ in Problem 2 is estimated from above by the value function $V(t, x)$ of the ordinary impulse control problem (6), with terminal functional $\boldsymbol{\varphi}$. The value function $V(t, x)$ is the solution to the HJB variational inequality (7). The latter allows calculating the feedback control strategy once $V(t, x)$ is calculated.

Remark 1: It is known [14], [15] that in the absence of uncertainty there exists an optimal control which is the sum of at most n individual impulses.

Since the information set $\mathscr{X}_1[t]$ is varying, so dis the terminal functional $\boldsymbol{\varphi}(x)$, and it is hardly possible to guarantee the number of impulses in the realized control.

However, by recalculating the information set only at selected instants $\tau_1, \dots, \tau_m$, one can guarantee the existence of feedback control with $m \cdot n$ impulses at most.

F. Higher-Order Generalized Controls

We now extend the formulation of the previous section: instead of allowing impulse controls of only delta-type we will consider generalized functions (distributions) of arbitrary order k which allow impulses of higher order described by higher derivatives of delta-functions. The mathematical theory of such functions can be found in [16], [17].

Instead of system (4) we consider

$$\begin{cases} \dot{x}_1(t) = 0, \\ \dot{x}_2(t) = B(t)u(t), \quad x_2(t_0) = 0. \end{cases} \quad (10)$$

Here the control $u(t)$ is chosen from the class $D^*_{k,m}[\alpha, \beta]$ of continuous linear functionals over the linear normed space $D_{k,m}[\alpha, \beta]$ which consists of k times differentiable functions $\varphi(t) : [\alpha, \beta] \to \mathbb{R}^m$ with support contained in $[\alpha, \beta]$, (see [16], [17]) . The norm ρ in $D_{k,m}[\alpha, \beta]$ is defined as

$$\rho[\varphi] = \max_{t\in[\alpha,\beta]} \gamma[\gamma_0(\varphi(t)), \gamma_1(\varphi'(t)), \dots, \gamma_k(\varphi^{(k)}(t))],$$

where γ_j, γ are finite-dimensional norms in spaces $\mathbb{R}^m$ and $\mathbb{R}^{k+1}$ respectively. The norm $\rho[\varphi]$ determines its adjoint norm $\rho^*[u]$ in the space $D^*_{k,m}[\alpha, \beta]$. Hence the control u is a distribution of order $k_u \le k$ which includes δ-functions and their higher derivatives of order up to k. Then trajectories $x_2(t)$ of the system (10) are distributions from $D^*_{k-1,n}[\alpha, \beta]$.

The admissible controls $u(t)$ are distributions from $D^*_{k,m}[\alpha, \beta]$ for which there exists a distribution $x_2(t) \in D^*_{k-1,n}[\alpha, \beta]$ which satisfies equation

$$\dot{x}_2(t) = B(t)u + f^{(\alpha)} - f^{(\beta)}$$

in the sense of distributions. The support of $x_2(t)$ is enclosed in $[t_0, t_1]$, where $[t_0, t_1] \subset [\alpha, \beta]$.

Here $f^{(\alpha)}$ and $f^{(\beta)}$ are distributions concentrated at points t_0 and t_1 respectively. They may be interpreted as initial and final conditions for the "trajectory" $x_2(t)$ and may be represented as

$$f^{(\alpha)} = \sum_{j=0}^{k} \alpha_j \delta^{(j)}(t - t_0), \quad f^{(\beta)} = \sum_{j=0}^{k} \beta_j \delta^{(j)}(t - t_1).$$

Recall that any distribution $u \in D^*_{k,m}[\alpha, \beta]$ may be written as (see [17])

$$\langle u, \varphi\rangle = \sum_{j=0}^{k} \int_\alpha^\beta (-1)^j \frac{d^j\varphi}{dt^j} dU_j(t), \quad (11)$$

where U_j are functions of bounded variation on $[\alpha, \beta]$, taking values in $\mathbb{R}^m$ and constant on $[\alpha, t_0) \cup (t_1, \beta]$.

Using the representation (11) one may see that the problem with generalized controls may be reduced to the problem 2 for the system with "ordinary" impulse controls:

$$\begin{cases} \dot{x}_1(t) = 0, \\ dx_2(t) = \mathbf{B}(t)dU(t), \quad x_2(t_0) = 0, \end{cases}$$

where matrix

$$\mathbf{B}(t) = \begin{pmatrix} B(t) & -B'(t) & B''(t) & \dots & (-1)^k B^{(k)}(t) \end{pmatrix}.$$

As indicated in [9], [10], for a completely controllable system the vector x may be moved from any point x^0 to any other point x^1 in zero time by control given as the sum of not more than n impulses $\delta^{(i)}$, $i = \overline{0, k}$. But such solutions do not have a physical interpretation. Their physically realizable bounded approximations are "fast" controls – piecewise constant functions concentrated on arbitrary small intervals. These functions may be selected such that x^0 may be moved to x^1 in arbitrary small "nano-time" (see [10] for details).

Our next step will be to indicate some computational tools. As mentioned above, the high order impulse control problem reduces to one with only "ordinary" impulses. Hence the next part is explained for only the "ordinary" case. To calculate $\boldsymbol{\varphi}^*(p)$, we use ellipsoidal approximations of the information set as described in the next subsection.

G. The Ellipsoidal Approximation

Here we assume that the sets $\mathscr{Q}(t)$ and $\mathscr{M}$ are ellipsoids[4] [18]: $\mathscr{Q}(t) = \mathscr{E}\,(q(t), Q(t))$, $\mathscr{M} = \mathscr{E}\,(m, M)$, with known parameters $q(t), m \in \mathbb{R}^n$ and $Q(t), M \in \mathbb{R}^{n\times n}$.

We will further substitute $\mathscr{X}_1[t]$ by its ellipsoidal approximation $\mathscr{Y}_+(t) = \mathscr{E}\,(\eta(t), Y(t))$. To find it we pass to a discrete-time analogue of (5) and then apply the formula for external approximation of intersection of two ellipsoids (see [19], [20]):

$$Y(t+\Delta t) = \alpha Z^{-1},$$
$$\eta(t+\Delta t) = Z^{-1}(\pi W_1 q_1 + (1-\pi) W_2 q_2),$$

where

$$Z = \pi W_1 + (1-\pi) W_2,$$
$$W_1 = Y^{-1}(t), \quad W_2 = H^T(t) Q^{-1}(t) H(t),$$
$$q_1 = \eta(t), \quad q_2 = H^{-1}(t)(y(t) - q(t)),$$
$$\alpha = 1 - \pi(1-\pi)\left\langle q_2 - q_1, W_2 Z^{-1} W_1 (q_2 - q_1)\right\rangle,$$

and parameter π is found numerically from the equation

$$\begin{aligned}\alpha(\det Z)^2 \operatorname{tr}(Z^{-1}(W_1 - W_2)) - \\ - \eta(\det Z)^2 \big(2\left\langle \eta(t+\Delta t), W_1 q_1 - W_2 q_2\right\rangle + \\ \left\langle \eta(t+\Delta t), (W_2 - W_1)\eta(t+\Delta t)\right\rangle - \\ - \left\langle q_1, W_1 q_1\right\rangle + \left\langle q_2, W_2 q_2\right\rangle\big) = 0.\end{aligned}$$

We then find the internal ellipsoidal approximation of the set $\mathscr{M}' = \mathscr{M} \dot{-} \mathscr{Y}_+(t)$. It is an ellipsoid $\mathscr{M}'_- = \mathscr{E}\,(m', M')$ with parameters

$$m' = m - \eta(t),$$
$$M' = \left(1 - \left(\frac{\langle \ell, M\ell\rangle}{\langle \ell, Y\ell\rangle}\right)^{\frac{1}{2}}\right) M + \left(1 - \left(\frac{\langle \ell, Y\ell\rangle}{\langle \ell, M\ell\rangle}\right)^{\frac{1}{2}}\right) Y,$$

where ℓ is a "good" direction [18].

Finally we use an ellipsoid $\mathscr{E}\,(m', M')$ as the target set to calculate the value function of the impulse control problem and the impulse feedback control.

[4]Ellipsoid $\mathscr{E}\,(r, R)$ with center $r \in \mathbb{R}^n$ and configuration matrix $R \geq 0$, $R \in \mathbb{R}^{n\times n}$ is a convex set with support function

$$\rho(p \mid \mathscr{E}\,(r, R)) = \langle p, r\rangle + \langle p, Rp\rangle^{\frac{1}{2}}.$$

If matrix R is non-degenerate, then

$$\mathscr{E}\,(r, R) = \{x \in \mathbb{R}^n \mid \left\langle x - r, R^{-1}(x - r)\right\rangle \leq 1\}.$$

Now the conjugate to the terminal function is the support function to set $\mathscr{M}'_-$ and is given by

$$\boldsymbol{\varphi}^*(p) = \langle p, m'\rangle + \langle p, M'p\rangle^{\frac{1}{2}}.$$

III. Output Feedback under Communication Constraints

We return to system (4) on time interval $[t_0, \vartheta]$, with measurements coming in discrete time due to equation

$$y(\tau_i) = H x_1(\tau_i) + \xi(\tau_i), \qquad i = \overline{1,k},$$

with $y(\tau_i), x_1(\tau_i) \in \mathbb{R}^n$, $t_0 \leq \tau_1 < \tau_2 < \ldots < \tau_k \leq \vartheta$, and noise $\xi(\tau_i)$ is for each i uniformly distributed on the set

$$\mathscr{Q} = \{\eta \in \mathbb{R}^n \mid |\eta_\ell| \leq \nu,\ \ell = \overline{1,n}\}.$$

The measurement signals are assumed to arrive at random time instants τ_i distributed according to Poisson[5] [21], with frequency λ.

Problem COM. Find a time interval of length $\vartheta - t_0$ and a control strategy $U = U(t, \mathscr{X}_1, x_2)$ restricted by condition $\operatorname*{Var}_{[t_0,\vartheta]} U(\cdot) \leq \mu$, such that the output feedback terminal control problem

$$\min\{\|x_1(\vartheta) + x_2(\vartheta)\| \mid U\} \leq \gamma \tag{12}$$

under given type of observations would be solvable with probability $P^0 \geq 1 - \varepsilon$, having γ, $\varepsilon > 0$ given in advance.

In order to ensure (12) we need to have the inclusion $x_2(\vartheta) + \mathscr{X}_1[\vartheta] \subseteq \gamma\mathscr{C}(0)$, which could be ensured by having $x_2(\vartheta) = -x_1^*$, $\mathscr{X}_1[\vartheta] \subseteq x_1^* + \mathscr{C}(0)$ for some x_1^*. Here $\mathscr{C}(0)$ is a unit cube in $\mathbb{R}^n$ with center 0.

Our Problem COM is thus reduced to two:

Problem I: find ϑ which ensures $\mathscr{X}_1[\vartheta] \subseteq \gamma\mathscr{C}(0) + x_1^*$ for some x_1^* with probability P^0, having γ, ε given, and

Problem II: ensure $x_2(\vartheta) = -x_1^*$.

With x_1^* and ϑ given, the second problem may be solved according to [9]. We therefore concentrate on Problem I.

Since the unknown $x_1(t) = c = \text{const}$, and $\mathscr{Q} = -\mathscr{Q}$, we have

$$c \in \bigcap\{H^{-1}(y(t_i) + \mathscr{Q}) \mid i = 1, \ldots, k\} = c^* + \mathscr{R}(k, \vartheta),$$
$$\mathscr{R}(k, \vartheta) = \bigcap\{H^{-1}(\xi^*(t_i) + \mathscr{Q}) \mid i = 1, \ldots, k\},\ \vartheta > t_k,$$

where $\xi^*(t_i)$ is the realization of $\xi(t_i)$ at the i-th measurement, $\mathscr{R}(k, \vartheta)$ is the measurement error after k measurements, and c^* is the realized value of c.

Our aim will now be to figure out with what probability we could have

$$\mathscr{R}(k, \vartheta) \subseteq \gamma\mathscr{C}(0). \tag{13}$$

The answer to this question is possible by direct calculation, using the reasoning of articles [21]. Indeed, the n-dimensional cube $\mathscr{Q}$ has 2^n vertices. Then, if the noise variable $\xi(t_i)$, $i = \overline{1,k}$, will happen to run around small neighborhoods $\mathscr{D}(v_m, \sigma)$ of all the vertices[6] v_m, $m = \overline{1, 2^n}$,

[5]The Poisson distribution is a standard tool for modeling communication signal transmission in discrete time.

[6]Here we consider n-dimensional cubes as neighborhoods $\mathscr{D}(v_m, \sigma)$ with center v_m and edges of length 2σ.

we will have $\mathscr{R}(k,\vartheta) \subset \mathscr{D}(0,\sigma)$ and the volume $V_{\mathscr{R}}(k,\vartheta)$ of $\mathscr{R}(k,\vartheta)$ will tend to zero with σ tending to zero

$$V_{\mathscr{R}}(k,\vartheta) \leq (2\sigma)^n \to 0.$$

In fact, the same result hold if the noise variable runs only through neighborhoods of $n+1$ vertices forming a simplex: $v_1 = (-\nu,-\nu,\ldots,-\nu)$, $v_2 = (\nu,-\nu,-\nu,\ldots,-\nu)$, $v_3 = (-\nu,\nu,-\nu,\ldots,-\nu)$, etc., $v_{n+1} = (-\nu,-\nu,\ldots,-\nu,\nu)$.

For each of those vertices v_j, $j = \overline{1,n+1}$, the probability that after k measurements the inclusion $\xi(t_i) \in \mathscr{R}(k,\vartheta) \cap \mathscr{D}(v_j,\sigma)$ is true at least once will be $P(\sigma,k,v_j) = 1-(1-\sigma^n\nu^{-n})^k$ and the probability that this would be true for all the vertices of $\mathscr{Q}$ is

$$P(\sigma,k) = (1-(1-\sigma^n\nu^{-n})^k)^{n+1}.$$

With n fixed, clearly $\lim_{k\to\infty} P(\sigma,k) = 1$ for any $\sigma > 0$.

Therefore, we now have to take $\sigma = \sigma^0$ small enough to ensure (13) and number $k = k^0$ which ensures

$$P(\sigma^0,k^0) \geq 1-\delta. \tag{14}$$

Then, clearly, we have $\mathscr{X}_1[\vartheta] \subseteq x_1^*(\vartheta) + \mathscr{D}(0,\sigma^0)$ for some $x_1^*(\vartheta)$ which can be determined from formula (a discrete version of formula given for $\rho(\ell \mid \mathscr{X}_1[\vartheta])$ given in subsection II-C)[7]

$$\rho(\ell \mid \mathscr{X}_1[\tau_k]) = \inf\Big\{ \sum_{i=1}^{k} \Big\langle \ell^{(i)}, y^*(\tau_i) \Big\rangle + \rho(-\ell^{(i)} \mid \mathscr{Q}) \;\Big|\; \ell^{(i)} \in \mathbb{R}^n, i = \overline{1,k}, \sum_{i=1}^{k} \ell^{(i)} = \ell \Big\}.$$

But to have such a number k^0, the interval of observations must be sufficiently large. The properties of the Poisson distribution with frequency λ indicate that the probability $P(k,\vartheta-t_0)$ of k measurements within interval $[t_0,\vartheta]$ is given by relation

$$P(k,\vartheta-t_0) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda(\vartheta-t_0))^j}{j!} \exp(-\lambda(\vartheta-t_0)).$$

Clearly, for any k we have

$$P(k,\vartheta-t_0) \to 1, \quad \vartheta \to \infty. \tag{15}$$

Summarizing the above we observe that the interval of observations $[t_0,\vartheta^0]$ which ensures $P^0 \geq 1-\varepsilon$ is determined from inequality

$$P^0 = P(k^0,\vartheta-t_0)P(\sigma^0,k^0) \geq P(k^0,\vartheta^0-t_0)(1-\delta) \geq 1-\varepsilon.$$

This inequality has a solution ϑ^0 for $\delta < \varepsilon$ (due to (15)).

Finally, for $\delta = \varepsilon/2$ we have ϑ^0 determined by relation

$$P(k^0,\vartheta^0-t_0) \geq 2(1-\varepsilon)(2-\varepsilon)^{-1}. \tag{16}$$

Theorem 2: Problem COM is solvable within an interval not less than $\vartheta^0 - t_0$ determined through (16) with k^0 found through (14) with restriction μ on control U large enough to ensure solvability of equation $x_2(\vartheta^0) = -x_1^*(\vartheta)$.

[7] Lower, in Remark 2 we indicate how to find $x_1^*(\vartheta)$ from these relations.

Theorem 3: For every starting position $\{t_0,x^0\}$ with initially unknown x^0 there exists a finite time $T = \vartheta^0 - t_0$ for which the origin $\{0\}$ is reachable exactly with probability 1 when μ is large enough.

Remark 2: Take the system of inequalities

$$\langle e^{(i)}, z\rangle \leq \rho(e^{(i)} \mid \mathscr{X}_1[\tau_k])$$

for all unit orths $\pm e^{(i)}$, $i = \overline{1,n}$. The solution

$$\mathscr{Z}(n,k) \supseteq \mathscr{X}_1[\vartheta], \quad \vartheta \geq \tau_k,$$

to this system is a rectangular polyhedron with center $z_1^* = z^*(n,k)$. It approximates $\mathscr{X}_1[\tau_k]$ from above and may be used as its substitute so that

$$\mathscr{Z}(n,k) - z^* \supseteq \mathscr{R}(k,\vartheta), \quad \vartheta \geq \tau_k$$

and $\mathscr{X}_1[\vartheta] \subseteq z^* + (\mathscr{Z}(n,k) - z^*) \subseteq z^* + \mathscr{D}(0,\sigma^0)$.

IV. Examples

A. Output Feedback Impulse Control

Here we present the results of numerical simulations for the system described below. The presentation will be accompanied by computer animation with comparative analysis of both approaches.

The problem is to design an impulse measurement feedback strategy to stop the oscillations of a rigidly suspended chain of N loaded springs by applying an impulse control force to a prescribed node of the chain. The chain also includes given loads attached in between the springs (the i-th load is attached to the lower end of the i-th spring). This is described by the following system of second-order ODEs:

$$\begin{cases} m_1\ddot{w}_1 &= k_2(w_2-w_1) - k_1w_1, \\ m_i\ddot{w}_i &= k_{i+1}(w_{i+1}-w_i) - k_i(w_i - w_{i-1}), \\ m_N\ddot{w}_N &= -k_N(w_N - w_{N-1}) + u, \end{cases}$$

when $t > t_0$. Here w_i is the displacement of the i-th load from the equilibrium, m_i is the mass of the i-th load, k_i is the stiffness coefficient of the i-th spring. The initial state at time t_0 is given by the displacements w_i^0 and the velocities of the loads $\dot{w}_i^0$.

The goal of the control is to steer the system to a neighborhood of the equilibrium in given finite time. We used the worst-case measurement ($\xi(t) \equiv 0$), leading to the largest possible information set.

In our numerical experiments we used the following values of parameters: $m_i = 1$, $k_i = 1$, $t_0 = 0$, $t_1 = 2\pi N$, $w_i^0 = \dot{w}_i^0 = 5$, $r(t) = 0$, $R(t) = \mathrm{diag}(10^{-4}I, 10^4I)$ (i.e. displacements are measured with relatively small error $\pm 0{,}01$ and velocities are measured with large error ± 100), $h = 2N$, $\Delta t = 0{,}1$ for ellipsoidal filter, $p(t) = 0$, $P(t) = 1$, $m = 0$, $M = I$.

In Figure 1 we show the dependence of diameter of the ellipsoidal information set on model time $t - t_0$. Note that this plot is the same for any size of chain N.

Figure 2 shows the realized impulse control for $N = 5$. Note that the dimension of the system here is $n = 2N = 10$, and the number of individual impulses (which is 13) here is greater than n.

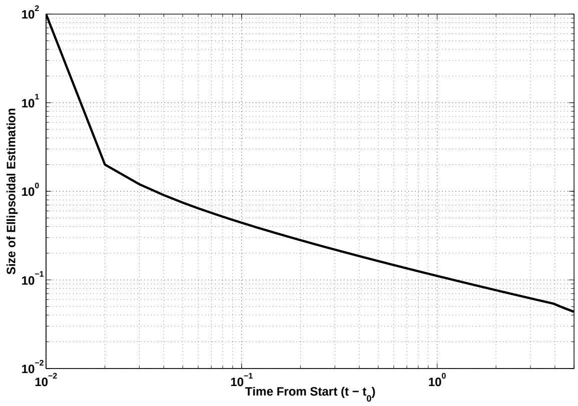

Fig. 1. Diameter of ellipsoidal information set versus time from start $(t - t_0)$

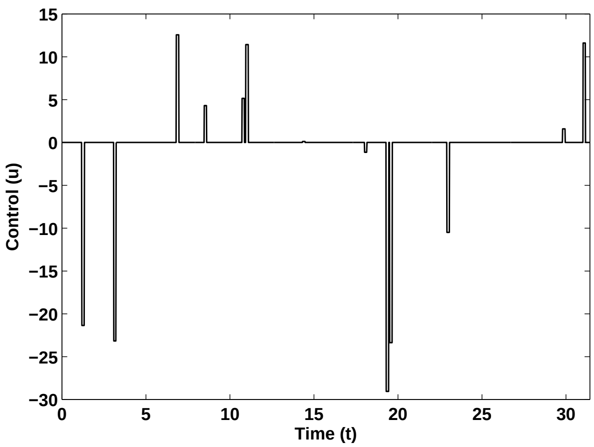

Fig. 2. Approximation of realized impulse control for $N = 5$

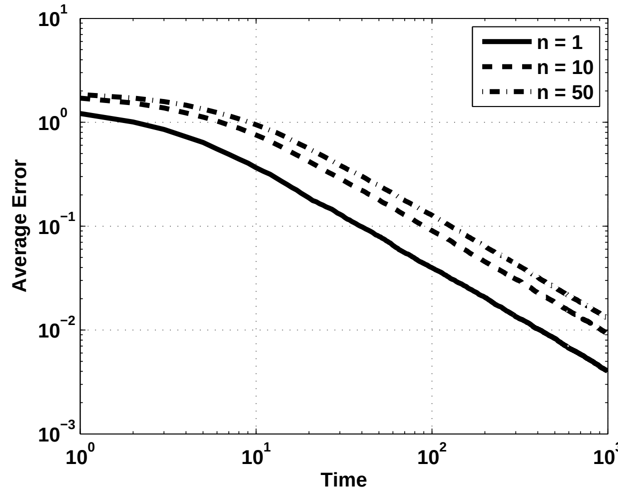

Fig. 3. Diameter of information set for the problem with Poisson measurements versus time from start $(\vartheta - t_0)$

B. Output Feedback under Communication Constraints

In Figure 3 we show the average size of the information set $\mathscr{R}(k, \vartheta)$ in the problem with communication constraints for various dimensions n.

V. Conclusions

This paper deals with *output feedback impulse control* under set-membership uncertainty where the control realization consists of a sequence of δ-impulses. It indicates solution schemes and suggests recommendation for computation. The problem is then generalized to the case of high-order "fast" controls which solve the terminal control problem in arbitrary small time. Finally an output feedback control problem is solved where communication signals for the available noisy measurements arrive at Poisson instants of time.

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