Output Feedback Strategies for Systems with Impulsive and Fast Controls

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Abstract

This paper deals with *output feedback impulse control* under set-membership uncertainty where the control realization consists of a sequence of δ-impulses. It indicates set-valued approaches for non-probabilistic continuous measurements and especially — for set-valued approach to stochastic case with discrete measurements.
The solution schemes are based on generalized Dynamic Programming relations of the HJB type. The problem is generalized to the case of high-order "fast" controls which solve the terminal control problem in arbitrary small time. Finally an output feedback control problem is solved where communication signals for the available noisy measurements arrive at Poisson instants of time. Numerical examples are demonstrated.

The Problem of Output Feedback Impulse Control

The **control system**:

$$dx(t) = A(t)x(t)dt + B(t)dU(t), \quad t \in [t_0, t_1]$$

- $x(t) \in \mathbb{R}^n$ — the state.
- $U(t) \in BV[t_0, t_1]$ — the impulse control.
- The terminal time t_1 is fixed.

Generalized Mayer–Bolza **functional**:

$$J(u(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(x(t_1 + 0)) \to \inf$$

- $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed and convex.

Measurement equation:

$$y(t) = H(t)x(t) + \xi(t).$$

Problem 1. *For a given terminal function $\varphi(\cdot)$, find a measurement impulse feedback control minimizing the functional $J(U(\cdot))$, despite the disturbance $\xi(\cdot)$, i.e. minimizing*

$$\mathscr{J}(U(\cdot)) = \max\{J(U(\cdot)) \mid x(\cdot)\},$$

over all trajectories consistent with available measurements.

The Information State

We decompose the system into two parts:

- the one with **uncertainty**
- the other with **impulses**.

$$x(t) = X(t,t_1)x_1(t) + X(t,t_1)x_2(t), \quad \begin{cases} \dot{x}_1(t) = 0, \\ dx_2(t) = B(t)dU(t), \quad x_2(t_0) = 0, \end{cases}$$

$X(t,\tau)$ — the transition matrix: $\partial X(t,\tau)/\partial t = A(t)X(t,\tau)$, $X(\tau,\tau) = I$.
The new **measurement equation** is

$$y_1(t) = H(t)X(t,t_1)x_1(t) + \xi(t) = y(t) - H(t)X(t,t_1)x_2(t).$$

Here we distinguish two problems —

- guaranteed state estimation
- feedback control in the space of states

(See A. B. Kurzhanski, "The problem of measurement feedback control," *Journal of Applied Mathematics and Mechanics*, vol. 68, no. 4, pp. 487–501, 2004.)

The **information state**: $\{t, \mathscr{X}_1[t], x_2(t)\}$
$\mathscr{X}_1[t]$ is the information set of all possible states $x_1(t)$ consistent with

- the system model
- the available measurement $y_1(s)$, $s \in [t_0, t]$, $t \le t_1$,
- the constraint $\mathscr{Q}$ on the unknown but bounded noise $\xi(\cdot)$.

$$\mathscr{X}_1[t] = \bigcap\{H^{-1}(t)\,(y_1^*(\tau) - \mathscr{Q}(\tau)) \mid \tau \in [t_0, t]\},$$

where realized measurements $y_1^*(t)$ on time interval $t \in [t_0, t]$ are given.

The information set may be also described through

- **support function**

$$\rho(\ell \mid \mathscr{X}_1[t]) = \inf_{\lambda(\cdot)} \left\{ \int_{t_0}^{t} \big(\langle \lambda(\tau), y_1^*(\tau)\rangle + +\rho(-\lambda(\tau) \mid \mathscr{Q}(\tau))\big) d\tau \mid \psi(t) = \ell \right\},$$

where the vector row ψ satisfies the ODE $\dot{\psi} = \lambda(t)H(t)$, $\psi(t_0) = 0$.

- **funnel equation** (here $h(A,B)$ is the Hausdorff distance between two compacts: $h(A,B) = \max\{h_+(A,B), h_-(A,B)\}$, $h_+(A,B) = \min\{\varepsilon \mid A \subseteq b + \varepsilon\mathscr{B}_1\}$, $h_-(A,B) = h_+(B,A)$):

$$\lim_{\sigma \to 0+0} \sigma^{-1} h\big(\mathscr{X}_1[t+\sigma], \mathscr{X}_1[t] \cap H^{-1}(t)(y_1^*(t) - \mathscr{Q}(t))\big) = 0.$$

(See A. B. Kurzhanski and T. F. Filippova, "On the theory of trajectory tubes: a mathematical formalism for uncertain dynamics, viability and control," in *Advances in Nonlinear Dynamics and Control*, ser. PSCT. Boston: Birkhäuser, 1993, no. 17, pp. 122–188.)

The Precise Formulation of the Problem

Problem 2. *Given position $\{t, \mathscr{X}_1, x_2\}$, $t \in [t_0, t_1]$, indicate an impulse feedback control strategy which minimizes the cost*

$$\mathscr{J}(U(\cdot)) = \operatorname*{Var}_{[t_0,t_1]} U(\cdot) + \varphi(\mathscr{X}_1[t_1] + x_2(t_1 + 0)),$$
$$\varphi(\mathscr{X}) = \max\{\varphi(x) \mid x \in \mathscr{X}\},$$

whatever be the measurement $y_1(t)$.

The Solution

Denote by $V(t, x; t_1, \varphi(\cdot))$ the **value function** in the impulse control problem with given terminal functional $\varphi(x)$:

$$V(t,x;t_1,\varphi(\cdot)) = \min_{U(\cdot)} \Big\{ \operatorname*{Var}_{[t,t_1]} U(\cdot) + \varphi(x(t_1+0)) \;\Big|\, x(t) = x,\ dx = B(\tau)dU \Big\}.$$

(See A. N. Daryin, A. B. Kurzhanski, and A. V. Seleznev, "A dynamic programming approach to the impulse control synthesis problem," in *Proc. Joint 44th IEEE CDC-ECC 2005*. Seville: IEEE, 2005.)

An estimate for the minimum terminal cost $\mathscr{V}(t, \mathscr{X}_1, x_2)$ in Problem 2:

$$\mathscr{V}(t, \mathscr{X}_1, x_2) \le V(t, 0; t_1, \boldsymbol{\varphi}(\cdot)), \quad \boldsymbol{\varphi}(x) = \max\{\varphi(x + z) \mid z \in \mathscr{X}_1 + x_2\}.$$

In particular, for $\varphi(x) = I(x \mid \mathscr{M})$: $\boldsymbol{\varphi}(x) = I(x \mid \mathscr{M} \dot{-} (\mathscr{X}_1 + x_2))$.

- $I(x \mid A)$ is the indicator function of the set A (0 in A, $+\infty$ outside of A).
- $\dot{-}$ is the geometric (Minkowski) difference between two convex sets: $A \dot{-} B = \{x \mid B + x \subseteq A\}$.

The value function $V(t, x; t_1, \varphi(\cdot))$ is the solution to the following variational inequality of **Hamilton–Jacobi–Bellman** type:

$$\min\{H_1(t,x,V_t,V_x),\ H_2(t,x,V_t,V_x)\} = 0,$$

with initial condition $V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot))$ and the Hamiltonians

$$H_1(t,x,\xi_t,\xi_x) = \xi_t, \quad H_2(t,x,\xi_t,\xi_x) = \min\{\langle \xi_x, B(t)u\rangle + 1 \mid \|u\| = 1\}.$$

In any position (t, x):

- either $H_1 = 0$, and the control may choose $dU(t) = 0$,
- or $H_1 > 0$, in which case it is necessary that $H_2 = 0$, and the control has a jump in direction $-B'(t)V_x$.

$V(t, x)$ may be computed through formula

$$V(t,x) = \sup[\langle p, x\rangle - \varphi^*(p) - I(p \mid \mathscr{B}_{\|\cdot\|_{[t,t_1]}}) \mid p \in \mathbb{R}^n] =$$
$$= \sup[\langle p, x\rangle - \varphi^*(p) \mid p \in \mathscr{B}_{\|\cdot\|_{[t,t_1]}}].$$

- φ^* is the Fenchel conjugate of φ
- $\mathscr{B}_{\|\cdot\|_{[[t,t_1]]}}$ is the unit ball in the vector norm $\|p\|_{[t,t_1]} = \left\|B^T(\cdot)p\right\|_{C[t,t_1]}$

Theorem 1. *The optimal value of the functional $\mathscr{J}(U(\cdot))$ in Problem 2 is estimated from above by the value function $V(t,x)$ of the ordinary impulse control problem with terminal functional $\boldsymbol{\varphi}$. The value function $V(t,x)$ is the solution to the HJB variational inequality.*

The Ellipsoidal Approximation

Ellipsoid $\mathscr{E}(r, R)$ is a convex set with support function

$$\rho(p \mid \mathscr{E}(r,R)) = \langle p, r\rangle + \langle p, Rp\rangle^{\frac{1}{2}}.$$

- center $r \in \mathbb{R}^n$
- configuration matrix $R \ge 0$, $R \in \mathbb{R}^{n \times n}$

If matrix R is non-degenerate, then

$$\mathscr{E}(r,R) = \{x \in \mathbb{R}^n \mid \left\langle x - r, R^{-1}(x - r)\right\rangle \le 1\}.$$

(See A. B. Kurzhanski and I. Vályi, *Ellipsoidal Calculus for Estimation and Control*, ser. SCFA. Boston: Birkhäuser, 1997.)

Assume the sets $\mathscr{Q}(t)$ and $\mathscr{M}$ to be known ellipsoids

$$\mathscr{Q}(t) = \mathscr{E}(q(t), Q(t)), \quad \mathscr{M} = \mathscr{E}(m, M),$$

Substitute $\mathscr{X}_1[t]$ by its ellipsoidal approximation $\mathscr{Y}_+(t) = \mathscr{E}(\eta(t), Y(t))$

- discrete-time analogue of the funnel equation
- external ellipsoidal approximation of intersection of two ellipsoids

(See L. Ros, A. Sabater, and F. Thomas, "An ellipsoidal calculus based on propagation and fusion," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 32, no. 4, 202.)

$$Y(t + \Delta t) = \alpha Z^{-1}, \quad \eta(t + \Delta t) = Z^{-1}(\pi W_1 q_1 + (1 - \pi)W_2 q_2),$$

where

$$Z = \pi W_1 + (1 - \pi)W_2, \quad W_1 = Y^{-1}(t), \quad W_2 = H^T(t)Q^{-1}(t)H(t),$$
$$q_1 = \eta(t), \quad q_2 = H^{-1}(t)(y(t) - q(t)),$$
$$\alpha = 1 - \pi(1 - \pi)\left\langle q_2 - q_1, W_2 Z^{-1} W_1 (q_2 - q_1)\right\rangle,$$

and parameter π is found numerically from the equation

$$\alpha(\det Z)^2 \operatorname{tr}(Z^{-1}(W_1 - W_2)) - \eta(\det Z)^2\big(2\langle \eta(t+\Delta t), W_1 q_1 - W_2 q_2\rangle +$$
$$\langle \eta(t+\Delta t), (W_2 - W_1)\eta(t+\Delta t)\rangle - \langle q_1, W_1 q_1\rangle + \langle q_2, W_2 q_2\rangle\big) = 0.$$

The internal approximation of $\mathscr{M}' = \mathscr{M} \dot{-} \mathscr{Y}_+(t)$ is $\mathscr{M}'_- = \mathscr{E}(m', M')$

$$m' = m - \eta(t),$$
$$M' = \left(1 - \left(\frac{\langle \ell, M\ell\rangle}{\langle \ell, Y\ell\rangle}\right)^{\frac{1}{2}}\right) M + \left(1 - \left(\frac{\langle \ell, Y\ell\rangle}{\langle \ell, M\ell\rangle}\right)^{\frac{1}{2}}\right) Y.$$

Finally we use an ellipsoid $\mathscr{E}(m', M')$ as the target set:

$$\boldsymbol{\varphi}^*(p) = \langle p, m'\rangle + \langle p, M'p\rangle^{\frac{1}{2}}.$$

Output Feedback under Communication Constraints

Suppose that measurements are coming in discrete time:

$$y(\tau_i) = Hx_1(\tau_i) + \xi(\tau_i), \quad i = \overline{1,k}, \quad t_0 \le \tau_1 < \tau_2 < \ldots < \tau_k \le \vartheta$$

and the noise $\xi(\tau_i)$ is uniformly distributed on the set

$$\mathscr{Q} = \{\eta \in \mathbb{R}^n \mid |\eta_\ell| \le \nu,\ \ell = \overline{1,n}\}.$$

The measurements arrive at Poisson time instants τ_i with frequency λ.

Problem COM. *Find a time interval of length $\vartheta - t_0$ and a control strategy $U = U(t, \mathscr{X}_1, x_2)$ restricted by condition $\operatorname{Var}_{[t_0,\vartheta]} U(\cdot) \le \mu$, such that the output feedback terminal control problem*

$$\min\{\|x_1(\vartheta) + x_2(\vartheta)\| \mid U\} \le \gamma$$

under given type of observations would be solvable with probability $P^0 \ge 1 - \varepsilon$, having γ, $\varepsilon > 0$ given in advance.

Problem COM is reduced to two problems:

- ***Problem I:*** find ϑ which ensures $\mathscr{X}_1[\vartheta] \subseteq \gamma\mathscr{C}(0) + x_1^*$ for some x_1^* with probability P^0, having γ, ε given ($\mathscr{C}(0)$ denotes a unit cube in $\mathbb{R}^n$ with center 0.)
- *Problem II:* ensure $x_2(\vartheta) = -x_1^*$.

Since the unknown $x_1(t) = c = \text{const}$, and $\mathscr{Q} = -\mathscr{Q}$, we have

$$c \in \bigcap\{H^{-1}(y(t_i) + \mathscr{Q}) \mid i = 1, \ldots, k\} = c^* + \mathscr{R}(k, \vartheta),$$
$$\mathscr{R}(k,\vartheta) = \bigcap\{H^{-1}(\xi^*(t_i) + \mathscr{Q}) \mid i = 1, \ldots, k\}, \ \vartheta > t_k,$$

where $\xi^*(t_i)$ is the realization of $\xi(t_i)$, $\mathscr{R}(k, \vartheta)$ is the measurement error after k measurements, and c^* is the realized value of c.

? — With what probability we could have $\mathscr{R}(k,\vartheta) \subseteq \gamma\mathscr{C}(0)$?
If $\xi(t_i)$ runs around small cubic neighborhoods $\mathscr{D}(v_m, \sigma)$ of all the 2^n vertices v_m, $m = \overline{1, 2^n}$, we will have $\mathscr{R}(k,\vartheta) \subset \mathscr{D}(0,\sigma)$ and the volume $V_{\mathscr{R}}(k,\vartheta)$ of $\mathscr{R}(k,\vartheta)$ will tend to zero with σ tending to zero

$$V_{\mathscr{R}}(k,\vartheta) \le (2\sigma)^n \to 0.$$

(See A. B. Kurzhanski, "Identification: a theory of guaranteed estimates," in *From Data to Model*, J. C. Willems, Ed. Springer, 1989, pp. 135–214.)

The same holds for any $n + 1$ vertices v_j forming a simplex. For each of v_j the probability that after k measurements the inclusion $\xi(t_i) \in \mathscr{R}(k,\vartheta) \cap \mathscr{D}(v_j,\sigma)$ is true at least once will be $P(\sigma, k, v_j) = 1 - (1 - \sigma^n \nu^{-n})^k$ and the probability that this would be true for all the vertices of $\mathscr{Q}$ is

$$P(\sigma, k) = (1 - (1 - \sigma^n \nu^{-n})^k)^{n+1} \to 1 \quad \forall \sigma > 0.$$

Take $\sigma = \sigma^0$ ensuring $\mathscr{R}(k,\vartheta) \subseteq \gamma\mathscr{C}(0)$ and number $k = k^0$ ensuring $P(\sigma^0, k^0) \ge 1 - \delta$. Then $\mathscr{X}_1[\vartheta] \subseteq x_1^*(\vartheta) + \mathscr{D}(0, \sigma^0)$ for some $x_1^*(\vartheta)$ determined from

$$\rho(\ell \mid \mathscr{X}_1[\tau_k]) = \inf\Big\{\sum_{i=1}^{k} \left\langle \ell^{(i)}, y^*(\tau_i)\right\rangle + \rho(-\ell^{(i)} \mid \mathscr{Q}) \;\Big|$$
$$\ell^{(i)} \in \mathbb{R}^n, i = \overline{1,k},\ \sum_{i=1}^{k} \ell^{(i)} = \ell\Big\}.$$

The probability $P(k, \vartheta - t_0)$ of k measurements within interval $[t_0, \vartheta]$ is

$$P(k, \vartheta - t_0) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda(\vartheta - t_0))^j}{j!} \exp(-\lambda(\vartheta - t_0)) \to 1 \quad \forall k.$$

The interval of observations $[t_0, \vartheta^0]$ ensuring $P^0 \ge 1 - \varepsilon$ is determined from inequality

$$P^0 = P(k^0, \vartheta - t_0)P(\sigma^0, k^0) \ge P(k^0, \vartheta^0 - t_0)(1 - \delta) \ge 1 - \varepsilon.$$

Finally, for $\delta = \varepsilon/2$ we have ϑ^0 determined by relation

$$P(k^0, \vartheta^0 - t_0) \ge 2(1 - \varepsilon)(2 - \varepsilon)^{-1}.$$

Theorem 2. *Problem COM is solvable within an interval not less than $\vartheta^0 - t_0$, with restriction μ on control U large enough to ensure solvability of equation $x_2(\vartheta^0) = -x_1^*(\vartheta)$.*

Theorem 3. *For every starting position $\{t_0, x^0\}$ with initially unknown x^0 there exists a finite time $T = \vartheta^0 - t_0$ for which the origin $\{0\}$ is reachable exactly with probability 1 when μ is large enough.*

Remark 1. *The scheme above is extendible to the case of time-varying matrices (for example, for periodic systems)*

$$H(t) = HX(t_0, t)x, \quad B(t) = X(t_0, t)B.$$

Numerical Experiments

Numerical algorithms are implemented using the **Ellipsoidal Toolbox**.
(See A. A. Kurzhanskiy and P. Varaiya, Ellipsoidal toolbox, `http://code.google.com/p/ellipsoids/`)
See **computer simulations** on the laptop.