Game-Theoretic Control under Different Classes of Restrictions on Pursuer and Evader

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Introduction

* Usually similar constraint classes are used for control and disturbances within the guaranteed approach:
  - ⇒ Hard Bounds (Geometric)
  - ⇒ Soft Bounds (Integral)

* In this work a combination of these constraints is considered:

Hard Bounds on Control	&	Soft Bounds on Disturbance

Problem Formulation

$$\dot{x}(t) \in \mathcal{U}\big(t, x(\cdot), u(\cdot)\big) + v(t), \qquad t_0 \le t \le t_1$$

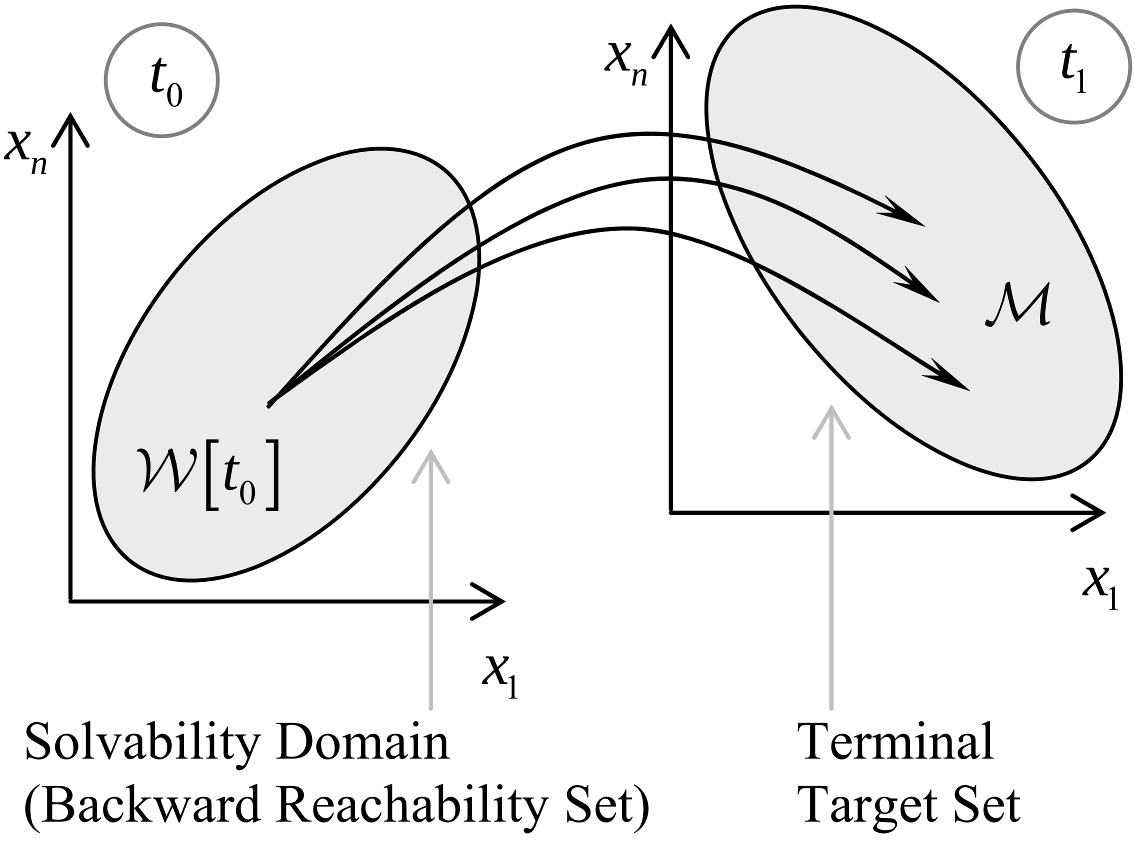
Control **is a closed-loop strategy with memory satisfying geometric constraint (*hard bounds*):**

$$\mathcal{U}\big(t, x(\cdot), u(\cdot)\big) \subseteq \mathcal{P}(t), \qquad t_0 \le t \le t_1$$

Disturbance **is a continuous function satisfying integral constraint (*soft bound*):**

$$\int_{t_0}^{t_1} \|v(t)\|^2 \, dt \le \mu^2$$

Problem I:



Equivalent Problem Formulation

$$\begin{cases} \dot{x}(t) \in \mathcal{U}\big(t, x(t), k(t)\big) + v(t), \\ \dot{k}(t) = -\|v(t)\|^2 \end{cases} \qquad t_0 \le t \le t_1$$

Control **is a closed strategy satisfying** ***geometric constraint*****:**

$$\mathcal{U}(t, x, k) \subseteq \mathcal{P}(t), \qquad t_0 \le t \le t_1$$

Disturbance **is a continuous function**
such that trajectories satisfy a ***state constraint*****:**

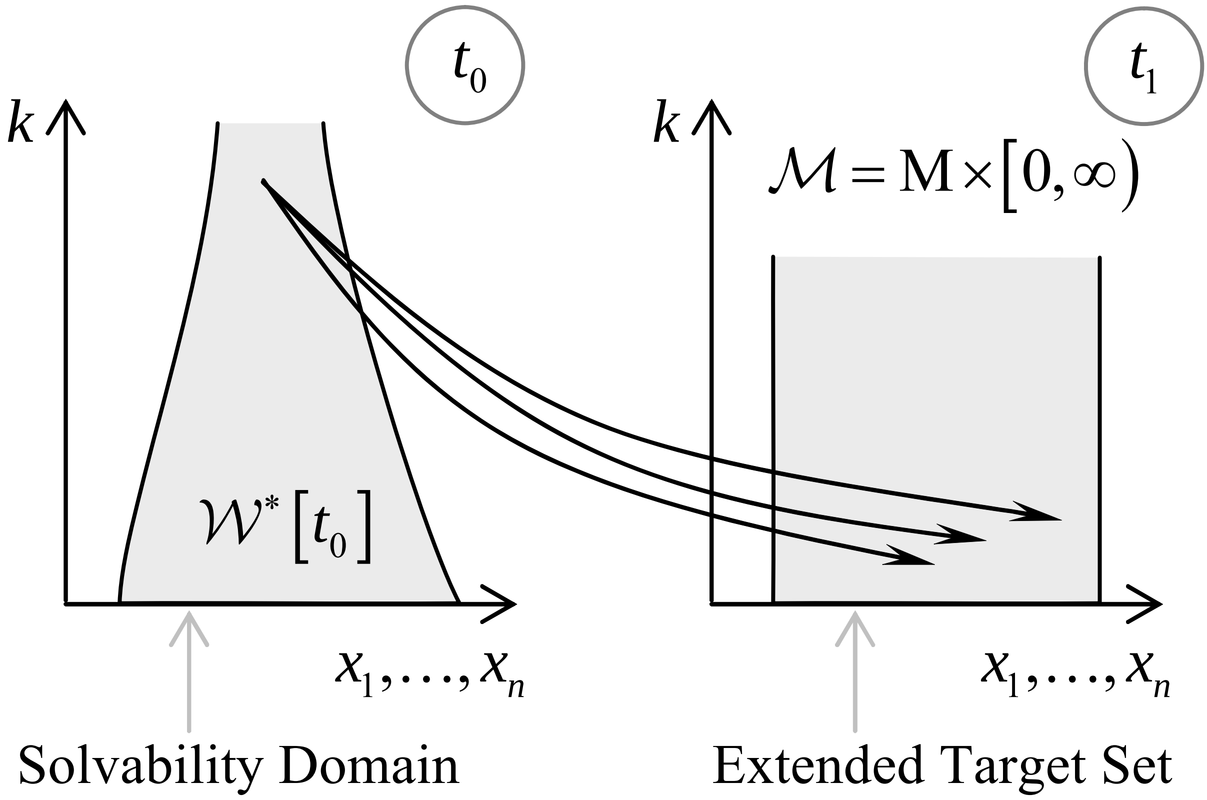
$$k(t_1) \ge 0 \quad \Leftrightarrow \quad \underbrace{k(t_0)}_{=\mu^2} - \int_{t_0}^{t_1} \|v(t)\|^2 \, dt \ge 0$$

The meaning of $k(t)$ **is** ***disturbance reserve*****.**

The current value of $k(t)$ ***is assumed to be known*** **to the control, since it can be calculated.**

Equivalent Problem Formulation (continued)

Problem II:

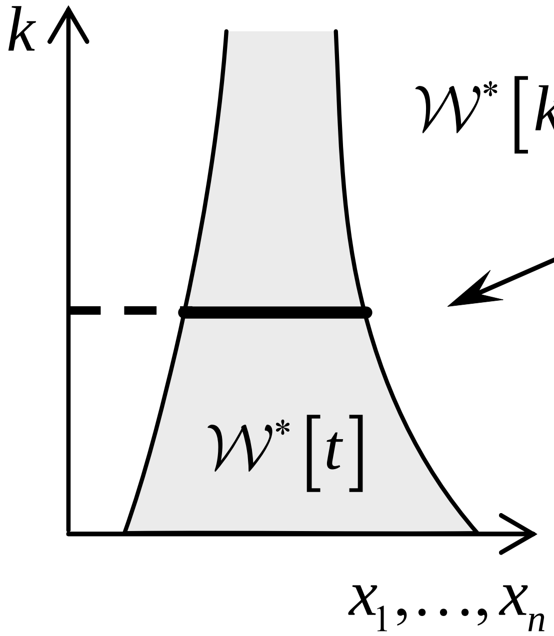


Problem II is *equivalent* to Problem I.

Solvability Domain and Terminal Target Set Sections

Solvability domain sections

$$\mathcal{W}^*[k,t] = \left\{ x \,\middle|\, (x,k) \in \mathcal{W}^*[t] \right\}$$



Solvability Domain is *not convex*

Its sections *are convex*

Terminal Target Set Sections

$$\mathcal{M}(k) = \left\{ x \,\middle|\, (x,k) \in \mathcal{M} \right\}$$

Assumptions on $\mathcal{M}(k)$:

1) $\mathcal{M}(k)$ is non-increasing, i.e.

$$k_1 \geq k_2 \quad \Rightarrow \quad \mathcal{M}(k_1) \subseteq \mathcal{M}(k_2).$$

2) $\mathcal{M}(k)$ is continuous.

3) Values of $\mathcal{M}(k)$ are convex compacts
(although $\mathcal{M}(k)$ itself may be nor convex nor compact)**.**

Dynamic Programming Approach

Value Function

$$V\left(t,x,k\right)=\min_{u(\cdot,\cdot,\cdot)}\max_{v(\cdot)}\left\{ d^2\left(x\left(t_1\right),\mathcal{M}\left(k\left(t_1\right)\right)\right) \;\middle|\; \begin{array}{c} k\left(t_1\right)\geq 0 \\ x\left(t\right)=x \\ k\left(t\right)=k \end{array} \right\}$$

$$\mathcal{W}^*\left[t,k\right]=\left\{x \;\middle|\; V\left(t,x,k\right)\leq 0\right\}$$

Hamilton-Jacobi-Bellman-Isaacs Equation:

$$\frac{\partial V}{\partial t}+\min_{u\in\mathcal{P}(t)}\max_{v\in\mathbb{R}^n}\left\{\left\langle\frac{\partial V}{\partial x},u+v\right\rangle-\left\|v\right\|^2\frac{\partial V}{\partial k}\right\}=0$$

$$\frac{\partial V}{\partial t}+\min_{u\in\mathcal{P}(t)}\left\langle\frac{\partial V}{\partial t},u\right\rangle\Bigg|_{k=0}=0$$

$$V\left(t_1,x,k\right)=d^2\left(x,\mathcal{M}\left(k\right)\right)$$

Semigroup Property (*Principle of Optimality*)

$$V\left(t,x,k;V\left(t_1,\cdot,\cdot\right)\right)=V\left(t,x,k;V\left(\tau,\cdot,\cdot;V\left(t_1,\cdot,\cdot\right)\right)\right)$$

$$V\left(t_1,x,k\right)=d^2\left(x,\mathcal{M}\left(k\right)\right)$$

Control Synthesis using the Value Function

$$\mathcal{U}^*\left(t,x,k\right)=\operatorname{Arg}\min\left\{\left\langle\frac{\partial V}{\partial x},u\right\rangle \;\middle|\; u\in\mathcal{P}\left(t\right)\right\}$$

Alternated Integral Construction

Max-min solvability domain for open-loop control strategies:

$$W^+[k,t] = \bigcap_{0\le\gamma\le k} \left(\mathcal{M}(\gamma) - \int_t^{t_1} \mathcal{P}(\tau)\,d\tau\right) \dot{-} \sqrt{(t_1-t)(k-\gamma)}B_1$$

(*) cf.: solvability domain in the case of geometric constraints (Pontryagin, 1980; Kurzhanski and Melnikov, 2000):

$$W^+[t] = \left(\mathcal{M} - \int_t^{t_1} \mathcal{P}(\tau)\,d\tau\right) \dot{-} \int_t^{t_1} \mathcal{Q}(\tau)\,d\tau$$

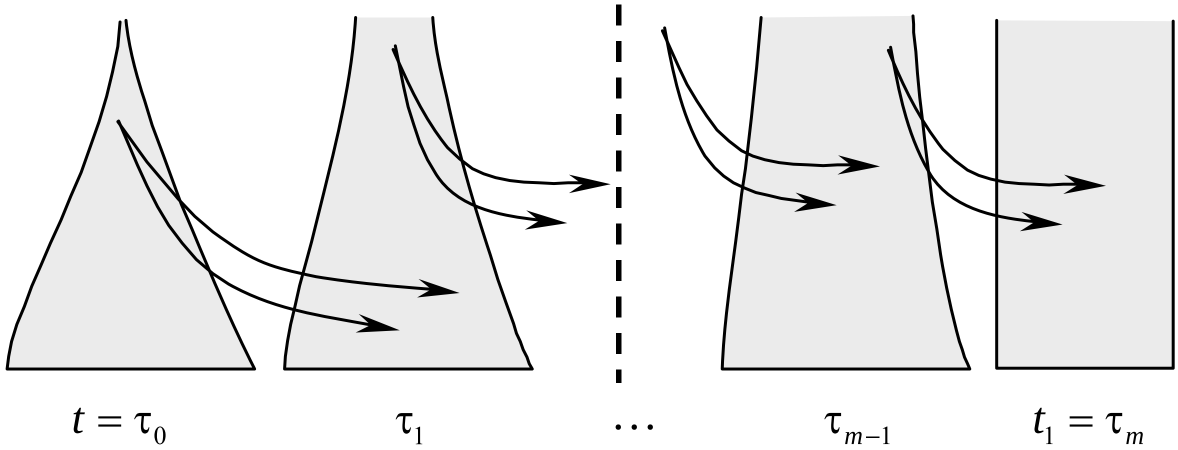
Upper integral sums:

$$\mathcal{T}: \quad t = \tau_0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m = t_1$$

$$\mathcal{I}^+(k,\tau_m,\mathcal{T}) = \mathcal{M}(k)$$

$$\mathcal{I}^+(k,\tau_{i-1},\mathcal{T}) = \bigcap_{0\le\gamma\le k} \left(\mathcal{I}^+(\gamma,\tau_i,\mathcal{T}) - \int_{\tau_{i-1}}^{\tau_i} \mathcal{P}(\tau)\,d\tau\right) \dot{-}$$

$$\dot{-}\sqrt{(\tau_i-\tau_{i-1})(k-\gamma)}B_1$$



Alternated Integral Construction (continued)

Upper Alternated Integral

$$\lim_{\operatorname{diam}\mathcal{T}\to 0} h\Big(\mathcal{I}^{+}\left(k,t,\mathcal{T}\right),\mathcal{I}^{+}\left(k,t\right)\Big)=0$$

Funnel Equation

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\max_{0\le\gamma\le k} h_{+}\left(\mathcal{I}^{+}\left(k,t\right)+\sqrt{\sigma\left(k-\gamma\right)}B_{1},\mathcal{I}^{+}\left(\gamma,t\right)-\sigma\mathcal{P}\left(t\right)\right)=0$$

Lower alternated integral is constructed the same way as upper one using the min-max open-loop solvability domains.

Alternated Integral:

$$\mathcal{I}\left(k,t\right)=\mathcal{I}^{+}\left(k,t\right)=\mathcal{I}^{-}\left(k,t\right)=\mathcal{W}^{*}\left[k,t\right]$$

Alternated Integral	Upper Alternated Integral	Lower Alternated Integral	Solvability Domain

Control Synthesis using the Extremal Aiming Rule

***Extremal Aiming Rule* (Krasovski, 1971) may be applied to find the control synthesis.**

Upper Estimate for the Value Function:

$$V\left(t,x,k\right) \leq d^2\left(x, \mathcal{W}^*\left[k,t\right]\right)$$

Full derivative of distance from the solvability domain:

$$H\left(t,x,k\right) = d^2\left(x, \mathcal{W}^*\left[t,k\right]\right)$$

or

$$H\left(t,x,k\right) = d^2\left(x, \mathcal{I}^-\left[t,k\right]\right)$$

$$\min_{u\in\mathcal{P}(t)} \max_{v\in\mathbb{R}^n} \frac{dH\left(t, x\left(t\right), k\left(t\right)\right)}{dt} =$$

$$= \frac{\partial H}{\partial t} + \min_{u\in\mathcal{P}(t)} \max_{v\in\mathbb{R}^n} \left\{ \left\langle \frac{\partial H}{\partial x}, u+v \right\rangle - \left\|v\right\|^2 \frac{\partial H}{\partial k} \right\} \leq 0$$

Control Synthesis:

$$\mathcal{U}^0\left(t,x,k\right) = \operatorname{Arg\,min}\left\{ \left\langle \frac{\partial H}{\partial x}, u \right\rangle \;\middle|\; u \in \mathcal{P}\left(t\right) \right\}$$

Another Combination of Constraints

$$\begin{pmatrix} \dot{x}(t) \\ \dot{k}(t) \end{pmatrix} \in \left\{ \begin{pmatrix} u+v \\ -\|u\|^2 \end{pmatrix} \;\middle|\; \begin{matrix} u \in \mathcal{U}\big(t, x(t), k(t)\big) \\ v \in \mathcal{Q}(t) \end{matrix} \right\}$$

Control **is a feedback strategy such that trajectories satisfy a** ***state constraint*****:**

$$k(t_1) \geq 0,$$

which is equivalent (for single-valued controls) to soft bound

$$\int_{t_0}^{t_1} \left\| u\big(t, x(t), k(t)\big) \right\|^2 dt \leq k(t_0)$$

Disturbance **satisfies geometric constraint:**

$$v(t) \in \mathcal{Q}(t)$$

Max-min solvability domain for open-loop control strategies:

$$W^{+}[k,t]=\bigcup_{0\le\gamma\le k}\left(\mathcal{M}(\gamma)-\sqrt{(t_1-t)(k-\gamma)}B_1\right)\dot{-}\int_t^{t_1}\mathcal{Q}(\tau)\,d\tau$$

Solvability domain funnel equation:

$$\lim_{\sigma\to 0}\frac{1}{\sigma}h\left(\mathcal{W}^*[k,t-\sigma]+\sigma\mathcal{Q}(t),\bigcup_{0\le\gamma\le k}\mathcal{W}^*[\gamma,t]-\sqrt{\sigma(k-\gamma)}B_1\right)=0$$

Hamilton-Jacobi-Bellman-Isaacs Equation:

$$\frac{\partial V}{\partial t}+\min_{u\in\mathbb{R}^n}\max_{v\in\mathcal{Q}(t)}\left\{\left\langle\frac{\partial V}{\partial x},u+v\right\rangle-\|u\|^2\frac{\partial V}{\partial k}\right\}=0$$

$$\left.\frac{\partial V}{\partial t}+\max_{v\in\mathcal{Q}(t)}\left\langle\frac{\partial V}{\partial x},v\right\rangle\right|_{k=0}=0$$

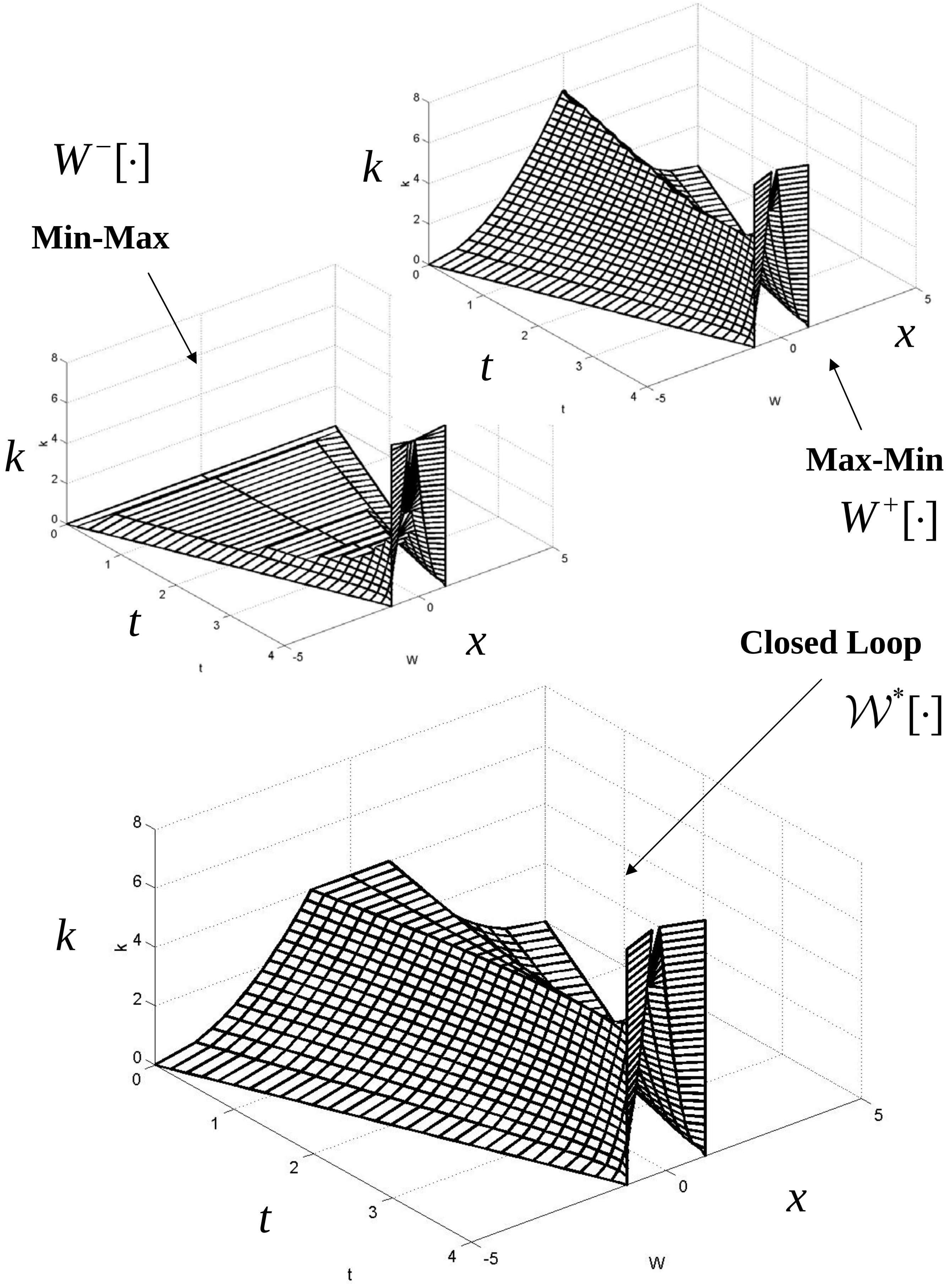
$$V(t_1,x,k)=d^2\left(x,\mathcal{M}(k)\right)$$

Control Synthesis:

$$\mathcal{U}^*(t,x,k)=\begin{cases}\mathbb{R}^n, & x\in\mathcal{W}^*[k,t],\\ \dfrac{1}{2}\left(\dfrac{\partial V}{\partial k}\right)^{-1}\dfrac{\partial V}{\partial x}, & x\notin\mathcal{W}^*[k,t]\end{cases}$$

$$V(t,x,k)=d^2\left(x,\mathcal{W}^*[k,t]\right)$$

1d Solvability Domain



Conclusion

1. Game-Theoretic Control Synthesis problem has been considered for an uncertain system where control and disturbance are chosen from *different classes*.

2. **Dynamic Programming** techniques can be applied to solve the problem, but this requires finding the solution of HJBI equation.

3. **Pontryagin's Alternated Integral** scheme can be adapted to find the solvability domain for this problem.

4. **Extremal Aiming Rule** gives the solution of control synthesis problem when the solvability tube is known.

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