NONLINEAR SYNTHESIS FOR UNCERTAIN SYSTEMS WITH DIVERSE TYPES OF CONSTRAINTS

Alexander Daryin [1]

Moscow State University
Faculty of Computational Mathematics and Cybernetics
Department of Systems Analysis
e-mail: daryin@pisem.net

Abstract: This work considers the problem of control synthesis for uncertain systems when control and disturbance are subject to geometric and integral constraints respectively (and vice versa). Solution is based on combination of dynamic programming techniques and convex analysis. An analogue of Pontryagin's alternated integral is developed, which is proved to be the solution of an evolution equation. Control synthesis is then built using extremal aiming rule. Hamilton–Jacobi–Bellman–Isaacs equation is obtained for the value function.

Keywords: control system synthesis, target control, uncertain dynamic systems, continuous time systems, closed-loop control, state feedback, constraints

1. INTRODUCTION

It is customary to consider that control and disturbance belong to classes of the same kind when dealing with the problem of control synthesis under uncertainty. For example, both are subject to geometric (Krasovski, 1971; Kurzhanski, 1999) or integral (Başar and Bernhard, 1995) constraints. However, in practical problems it is not always the case. We shall consider a problem of control synthesis under uncertainty with geometric constraints on control and integral constraints on disturbance (for reverse situation a brief formulation of results will be given). While solving this problem, we follow the lines of the article by Kurzhanski and Melnikov (2000), but with different types of constraints.

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2. PROBLEM FORMULATION

The system under consideration is

$$\dot{x}(t) \in A(t)x(t) + B(t)\mathcal{U}(t, x(\cdot), u(\cdot)) + \\ + C(t)v(t), \quad t \in T = [t_0, t_1] \quad (1)$$

with continuous matrix functions $A(t)$, $B(t)$ and $C(t)$, the latter also being injective for all $t \in T$. Here $x(t) \in \mathbb{R}^n$ is the state space vector, $\mathcal{U}(t) = \mathcal{U}(t, x(\cdot), u(\cdot)) \subseteq \mathbb{R}^p$ is the nonlinear control strategy with memory, and $v(t) \in \mathbb{R}^q$ is the noise input. The control input is restricted for almost all $t \in T$ by a geometric constraint

$$\mathcal{U}(t) \subseteq \mathcal{P}(t) \quad (2)$$

where $\mathcal{P}(\cdot)$ is a given multivalued mapping with nonempty convex compact values continuous in the Hausdorff metric. The disturbance $v(\cdot)$ is a Lebesgue-measurable function satisfying quadratic integral constraint

$$\int_{t_0}^{t_1} \langle v(t), Q(t)v(t) \rangle \, dt \leqslant \nu, \quad (3)$$

where $Q(t)$ is a positively defined matrix function continuous on the segment $T = [t_0, t_1]$.

Two classes of controls will be considered further:

(1) *closed-loop strategies with memory* ($U_{\mathrm{CL,M}}$): multivalued functions $\mathcal{U}(t, x(\cdot), u(\cdot)) \subseteq \mathcal{P}(t)$ assuring the existence and extendability of solutions of the differential inclusion (1) whatever continuous function $v(\cdot)$ is chosen, and
(2) *open-loop controls* (U_{OL}): measurable vector functions $u = u(t)$ satisfying (2). This class is only necessary for intermediate constructions.

Remark 1. Within the class of *closed loop* controls *without memory* $\mathcal{U}(t, x)$ the semigroup property of solvability tubes (i.e., the *principle of optimality*) is not satisfied, that's why controls with memory should by used.

Problem 1. Given a convex compact *terminal target set* M, specify a *solvability set* $\mathcal{W}(t_0, t_1, M) = \mathcal{W}[t_0]$ and a *closed loop control strategy* $\mathcal{U} \in U_{\mathrm{CL,M}}$ such that all solutions of the differential inclusion (1) that start from any position $(t_0, x) \in T \times \mathcal{W}[t_0]$ would reach the set M regardless of any disturbance $v(\cdot)$ restricted by (3).

2.1 *Equivalent problem*

System (1) can be rewritten in a simpler form

$$\dot{x}(t) \in \mathcal{U}(t, x(\cdot), u(\cdot)) + v(t), \quad t \in T = [t_0, t_1] \quad (4)$$

with new constraints

$$\mathcal{U}(t) \subseteq \mathcal{P}_0(t) \quad (5)$$

and

$$\int_{t_0}^{t_1} \langle v(t), Q_0(t) v(t) \rangle \, \mathrm{d}t \leqslant \nu \quad (6)$$

where

$$\mathcal{P}_0(t) = S(t_1, t) B(t) \mathcal{P}(t),$$
$$Q_0(t) = S(t_1, t) (C(t) Q^{-1}(t) C'(t))^{-1} S(t, t_1),$$

and $S(\tau, t)$ stands for the matrix solution of the equation

$$\frac{\partial S(t, \tau)}{\partial t} = A(t) S(t, \tau), \quad S(\tau, \tau) = I.$$

Terminal target set M remains the same after this change of variables.

Without loss of generality, system (4) with constraints (5), (6) will be further considered, omitting zero indices.

Problem 1 can be simplified by introducing additional state variable $k(t)$ by means of the following differential equation:

$$\dot{k}(t) = -\langle v(t), Q(t) v(t) \rangle, \quad k(t_0) = \nu. \quad (7)$$

Current value of this variable is assumed known to the control because it can be deduced from state and control trajectories.

Now integral constraint (3) can be recasted as a state constraint

$$k(t) \geqslant 0, \quad t_0 \leqslant t \leqslant t_1, \quad (8)$$

and it is easy to prove that problem 1 is equivalent to the following one (when $\mathcal{M} = M \times [0, \infty)$):

Problem 2. Given the *extended terminal set* $\mathcal{M} \subseteq \mathbb{R}^{n+1}$, specify a *solvability set* $\mathcal{W}^*(t, t_1, \mathcal{M}) = \mathcal{W}^*[t]$ and a *state feedback strategy* $\mathcal{U}(t, x, k) \in U_{\mathrm{CL}}$ such that all solutions of the differential inclusion

$$\begin{pmatrix} \dot{x}(t) \\ \dot{k}(t) \end{pmatrix} \in \begin{pmatrix} \mathcal{U}(t, x(t), k(t)) + v(t) \\ -\langle v(t), Q(t) v(t) \rangle \end{pmatrix} \quad (9)$$

that start from any position $(t_0, x, k) \in T \times \mathcal{W}^*[t_0]$ would reach the set $\mathcal{M}$ regardless of any disturbance $v(\cdot)$ which ensures the fulfillment of state constraint (8).

Even though initial system (1) was linear, extended system in the problem 2 is nonlinear because of the equation (7), and moreover it has a state constraint (8). This results in solvability domains not being convex even in the most simple cases (fig. 1, 2). To have a possibility of applying convex analysis techniques, instead of terminal target sets and solvability domains themselves their sections at constant k levels will be considered:

$$\mathcal{M}(k) = \{x \mid (x, k) \in \mathcal{M}\}, \quad (10)$$
$$\mathcal{W}^*[k, t] = \{x \mid (x, k) \in \mathcal{W}^*[t]\} \quad (11)$$

These sections are, on the contrary, always convex. It will be assumed that multivalued mapping $\mathcal{M}(k)$ has compact convex values and it is non-increasing, i.e.

$$k_1 > k_2 \quad \Rightarrow \quad \mathcal{M}(k_1) \subseteq \mathcal{M}(k_2).$$

It will be shown further that the key element in the solution of the control synthesis problem 2 is the multivalued mapping $\mathcal{W}^*[t]$ which can be computed using an analogue of *Pontryagin's alternated integral* (Pontryagin, 1980; Varaiya and Lin, 1969; Nikolski, 1985). Once this mapping is found, control synthesis is obtained by applying the *extremal aiming rule* (Krasovski, 1971; Krasovski and Subbotin, 1988). Ellipsoidal calculus developed by Kurzhanski and Vályi (1997) can be used to find inner approximation of the solvability domain and to obtain control synthesis in form of an "analytical regulator", i.e. in closed form.

3. SEQUENTIAL MAXIMIN AND MINIMAX

Definition 2. Maximin solvability domain $W^+[t]$ (within the class of open-loop controls) is the set of all positions (x, k) such that for any disturbance input $v(\cdot)$ satisfying (8) there exists an open-loop control $u(\cdot)$ ensuring $x(t_1) \in \mathcal{M}(k(t_1))$.

Definition 3. Minimax solvability domain $W^-[t]$ (within the class of open-loop controls) is the set of all positions (x, k) such that there exists an open-loop control $u(\cdot)$ ensuring $x(t_1) \in \mathcal{M}(k(t_1))$ regardless of any disturbance input $v(\cdot)$ satisfying (8).

Lemma 4. The following formulae are true for the sections of maximin and minimax solvability domains:

$$W^+[k,t] = W^+(k,t,t_1,\mathcal{M}(\cdot)) = \\ = \bigcap_{0\leqslant\gamma\leqslant k}\left(\mathcal{M}(\gamma) - \int_t^{t_1}\mathcal{P}(\tau)\,\mathrm{d}\tau\right) \dot{-} \\ \dot{-}\sqrt{k-\gamma}\,\mathcal{Q}(t,t_1), \quad (12)$$

$$W^-[k,t] = \bigcap_{0\leqslant\gamma\leqslant k}\mathcal{M}(\gamma) \dot{-} \sqrt{k-\gamma}\,\mathcal{Q}(t,t_1) - \\ - \int_t^{t_1}\mathcal{P}(\tau)\,\mathrm{d}\tau \quad (13)$$

where $\mathcal{Q}(t,t_1)$ is an ellipsoid

$$\mathcal{Q}(t,t_1) = \mathcal{E}\left(0, \int_t^{t_1} Q^{-1}(\tau)\,d\tau\right) = \\ = \left\{x \;\middle|\; \left\langle x, \left(\int_t^{t_1} Q^{-1}(\tau)\,d\tau\right)^{-1} x\right\rangle \leqslant 1\right\}; \quad (14)$$

in particular, when $Q(t) \equiv 1$, then

$$\mathcal{Q}(t,t_1) = \sqrt{t_1 - t}B, \quad B = \{x \mid \|x\| \leqslant 1\}.$$

Corollary 5. When $\mathcal{M} = M \times [0,\infty)$, i.e. $\mathcal{M}(k) \equiv M$, expressions (12) and (13) take up the following form:

$$W^-[k,t] = \left(M - \int_t^{t_1}\mathcal{P}(\tau)\,\mathrm{d}\tau\right) \dot{-} \sqrt{k}\,\mathcal{Q}(t,t_1),$$

$$W^-[k,t] = M \dot{-} \sqrt{k}\,\mathcal{Q}(t,t_1) - \int_t^{t_1}\mathcal{P}(\tau)\,\mathrm{d}\tau.$$

Now, using these open-loop solvability sets upper and lower alternated integrals will be constructed. Let $\mathcal{T} = \{\tau_0, \ldots, \tau_m\}$ be an arbitrary subdivision of interval $[t, t_1]$, where $\tau_i - \tau_{i-1} = \sigma_i > 0$. As a first step at instant t_1 let

$$W^+_{\mathcal{T}}[k,\tau_m] = \mathcal{M}(k). \quad (15)$$

Then, at each step find the open-loop solvability set

$$W^+_{\mathcal{T}}[k,\tau_{i-1}] = W^+(k,\tau_{i-1},\tau_i,W^+_{\mathcal{T}}[\cdot,t]), \quad (16)$$

which due to (12) gives

$$W^+[k,\tau_{i-1}] = \bigcap_{0\leqslant\gamma\leqslant k}\left(W^+_{\mathcal{T}}[\gamma,\tau_i] - \\ - \int_{\tau_{i-1}}^{\tau_i}\mathcal{P}(\tau)\,\mathrm{d}\tau\right) \dot{-} \sqrt{k-\gamma}\,\mathcal{Q}(\tau_{i-1},\tau_i). \quad (17)$$

Finally, the value

$$W^+_{\mathcal{T}}[k,\tau_0] = \mathcal{I}^+(k,t,t_1,\mathcal{M}(\cdot),\mathcal{T}) \quad (18)$$

is called the *upper integral sum* corresponding to the subdivision $\mathcal{T}$. The latter is the solvability set for the *motion correction problem* when at each of the instants τ_i the current position $(x(\tau_i), k(\tau_i))$ and disturbance values $v(\cdot)$ on the interval $[\tau_i, \tau_{i+1}]$ are reported to the control.

Definition 6. If there exists a Hausdorff limit $\mathcal{I}^+[k,t] = \mathcal{I}^+(k,t,t_1,\mathcal{M}(\cdot))$ of upper integral sums when $\max\{\sigma_i\} \to 0$, it is called the *upper alternated integral.*

Lower integral sums $\mathcal{I}^-(k,t,t_1,\mathcal{M}(\cdot),\mathcal{T})$ and *lower alternated integral* $\mathcal{I}^-[k,t]$ are constructed the same way as upper alternated integral using minimax solvability sets in the class of open-loop controls.

If both lower and upper integral exist and they coincide, the set

$$\mathcal{I}[k,t] = \mathcal{I}^+[k,t] = \mathcal{I}^-[k,t] \quad (19)$$

is then called the *alternated integral*, or *alternated solvability set* of the problem 2.

Lemma 7. Once $\mathcal{I}^+[\nu,t_0] \neq \varnothing$, the multivalued function $\mathcal{I}^+[k,t]$ is the maximum solution of the evolution equation (for $t \in T$, $k \in [0,\nu]$)

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\max_{0\leqslant\gamma\leqslant k} \\ h_+\Big(\mathcal{I}^+[k,t-\sigma] + \sqrt{\sigma(k-\gamma)}\,\mathcal{E}\left(0,Q^{-1}(t)\right), \\ \mathcal{I}^+[\gamma,t] - \sigma\mathcal{P}(t)\Big) = 0. \quad (20)$$

Lemma 8. If both upper and lower alternated integrals exist, the following inclusion is true:

$$\mathcal{I}^-[k,t] \subseteq \mathcal{W}^*[k,t] \subseteq \mathcal{I}^+[k,t]. \quad (21)$$

Theorem 9. (1) If the alternated integral exists, it coincides with the solvability domain of problem 2:

$$\mathcal{I}[k,t] = \mathcal{W}^*[k,t]. \quad (22)$$

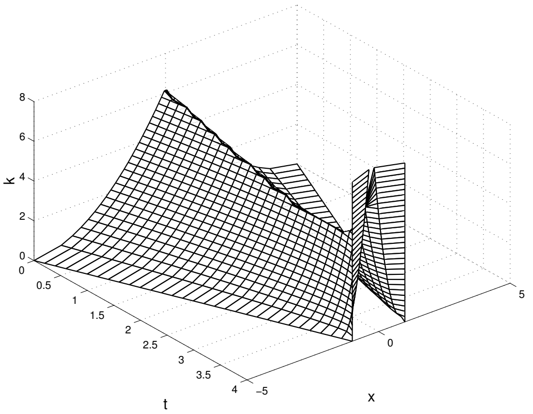

Fig. 1. Maximin Solvability Domain Boundary for One-Dimensional Case

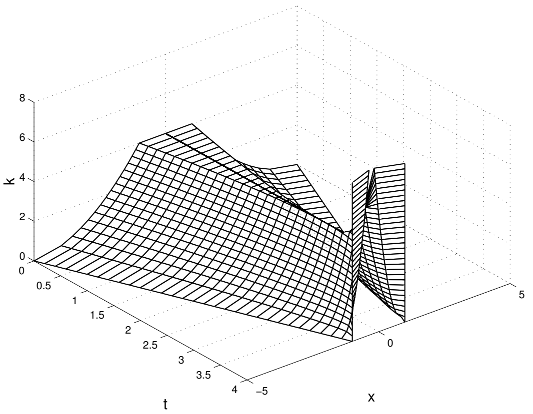

Fig. 2. Solvability Domain Boundary for One-Dimensional Case

(2) Values of alternated integral satisfy the semigroup property in backward time (that is, the *principle of optimality*):

$$\begin{aligned}&\mathcal{I}(k, t, t_1, \mathcal{M}(\cdot)) = \\ &\quad = \mathcal{I}(k, t, \tau, \mathcal{I}(\cdot, \tau, t_1, \mathcal{M}(\cdot))).\end{aligned} \qquad (23)$$

4. CONTROL SYNTHESIS

4.1 *Value Function*

The problem 2 does not contain any optimization criterion: it is only necessary to find some "guaranteeing" solution. Nevertheless, this problem can be posed as an extremal one and solved using dynamic programming techniques. With this aim in view, consider the following *value function*:

$$\begin{aligned}V(t, x, k) = &\min_{\mathcal{U} \in U_{\mathrm{CL}}} \max_{v(\cdot)} \\ &\max_{(x[t_1], k[t_1]) \in X[t_1]} d^2(x[t_1], \mathcal{M}(k[t_1])),\end{aligned} \qquad (24)$$

where $X[t]$ is the attainability tube of the differential inclusion (9).

Lemma 10. Value function (24) and solvability set $\mathcal{W}^*[k, t]$ are bound by the following statements:

(1) solvability domain is the level set of value function:

$$\mathcal{W}^*[k, t] = \{x \mid V(t, x, k) \leqslant 0\}; \qquad (25)$$

(2) the following estimation holds when solvability set is not empty:

$$V(t, x, k) \leqslant d^2(x, \mathcal{W}^*[k, t]), \qquad (26)$$

which means that under optimal control the distance between trajectory endpoint and terminal target set $\mathcal{M}(k(t_1))$ is equal to or smaller than the distance between initial point and solvability set $\mathcal{W}^*[k, t]$.

4.2 *Control Synthesis Using Value Function*

Theorem 11. (1) The value function satisfies the *principle of optimality*:

$$\begin{aligned}&V(t, x, k; V(t_1, \cdot, \cdot)) = \\ &\quad = V(t, x, k; V(\tau, \cdot, \cdot; V(t_1, \cdot, \cdot))),\end{aligned} \qquad (27)$$

where $V(t_1, x, k) = d^2(x, \mathcal{M}(k))$.

(2) The value function is the solution of Hamilton–Jacobi–Bellman–Isaacs (H-J-B-I) equation

$$\begin{aligned}&\frac{\partial V}{\partial t} + \min_{u \in \mathcal{P}(t)} \max_{v \in \mathbb{R}^n} \left\{ \left\langle \frac{\partial V}{\partial x}, u + v \right\rangle - \right. \\ &\qquad \left. - \langle v, Q(t) v \rangle \frac{\partial V}{\partial k} \right\} = 0\end{aligned} \qquad (28)$$

when $t_0 \leqslant t \leqslant t_1$, $k > 0$, with the boundary condition

$$\frac{\partial V}{\partial t} + \min_{u \in \mathcal{P}(t)} \left\langle \frac{\partial V}{\partial x}, u \right\rangle \Big|_{k=0} = 0 \qquad (29)$$

and initial condition

$$V(t_1, x, k) = d^2(x, \mathcal{M}(k)). \qquad (30)$$

(3) Optimal control strategy is expressed as

$$\mathcal{U}^*(t, x, k) = \operatorname*{Arg\,min}_{u \in \mathcal{P}(t)} \left\langle \frac{\partial V(t, x, k)}{\partial x}, u \right\rangle. \qquad (31)$$

To compute this control strategy it is not necessary to know the value function itself, but its level sets will be enough because expression (31) uses only the gradient of the value function. This idea is developed in the next paragraph.

4.3 *Control Synthesis Using Solvability Domain*

Though expression (31) gives the optimal control strategy, it is difficult to use because of having to solve the H-J-B-I equation (27). However, there exists another way of control synthesis for problem 2 which consists in applying the "extremal aiming rule":

Theorem 12. Consider the function

$$H(t, x, k) = d^2(x, \mathcal{W}^*[k, t]) \tag{32}$$

or (if alternated integral does not exist)

$$H(t, x, k) = d^2(x, \mathcal{I}^-[k, t]). \tag{33}$$

The following assertions hold:

(1) The total derivative of function $H(t, x, k)$ along trajectories of system (4), (7) meets the following inequality:

$$\min_{u \in \mathcal{P}(t)} \max_{v \in \mathbb{R}^n} \frac{\mathrm{d}H(t, x(t), k(t)}{\mathrm{d}t} =$$
$$= \frac{\partial H}{\partial t} + \min_{u \in \mathcal{P}(t)} \max_{v \in \mathbb{R}^n} \left\{ \left\langle \frac{\partial H}{\partial x}, u + v \right\rangle - \right.$$
$$\left. - \langle v, Q(t) v \rangle \frac{\partial H}{\partial k} \right\} \leqslant 0. \tag{34}$$

(2) Feedback control strategy

$$\mathcal{U}^0(t, x, k) = \operatorname*{Arg\,min}_{u \in \mathcal{P}(t)} \left\langle \frac{\partial H(t, x, k)}{\partial x}, u \right\rangle \tag{35}$$

solves the problem (2).

Note that though control strategy (35) does not minimize the distance between $x(t_1)$ and $\mathcal{M}(k(t_1))$, it guarantees that the latter will not be greater than the distance between trajectory start point and solvability set, as does optimal control $\mathcal{U}^*(t, x, k)$.

5. ANOTHER TYPE OF CONSTRAINTS

Let us now consider the situation with reverse types of constraints: integral for control inputs and geometric for disturbance. Equivalent problem will be considered straight away:

$$\begin{cases} \dot{x}(t) &= u(t) + v(t), \\ \dot{k}(t) &= -\langle u(t), P(t) u(t) \rangle, \end{cases} \quad t \in [t_0, t_1], \tag{36}$$

where, as before, $(x(t), k(t))$ is the state space vector, $u(t)$ is the control input and $v(t)$ is the noise input. $P(t)$ is a positively defined matrix function continuous on the segment $T = [t_0, t_1]$.

Control input is Lebesgue-integrable function such that trajectories of system satisfy the state constraint

$$k(t) \geqslant 0, \quad t \in T, \tag{37}$$

which is identical to integral quadratic contraint

$$\int_{t_0}^{t_1} \langle u(t), P(t) u(t) \rangle \, \mathrm{d}t \leqslant k(t_0). \tag{38}$$

Noise input is a Lebesgue-measurable function restricted by geometric constraint

$$v(t) \in \mathcal{Q}(t), \quad t \in T, \tag{39}$$

where set-valued mapping $\mathcal{Q}(t)$ is continuous in Hausdorff metric.

As before, control strategy can belong to one of the two classes:

(1) *closed-loop* (state feedback) strategies (U_{CL}): set-valued functions $\mathcal{U}(t, x, k)$ assuring the existence and extendability of solutions of the differential inclusion

$$\begin{pmatrix} \dot{x} \\ \dot{k} \end{pmatrix} \in \left\{ \begin{pmatrix} u + \mathcal{Q}(t) \\ -\langle u, Pu \rangle \end{pmatrix} \,\middle|\, u \in \mathcal{U}(t, x(t), k(t)) \right\}, \tag{40}$$

and

(2) *open-loop* strategies (U_{OL}): measurable functions $u = u(t)$ satisfying (2).

Remark 13. For a state feedback strategy $\mathcal{U} \in U_{\mathrm{CL}}$ to satisfy (37), it is sufficient that

$$\mathcal{U}(t, x, k) = \{0\}. \tag{41}$$

when $k < 0$.

Problem 3. Given the *terminal set* $\mathcal{M} \subseteq \mathbb{R}^{n+1}$, specify a *solvability set* $\mathcal{W}^*(t, t_1, \mathcal{M}) = \mathcal{W}^*[t]$ and a state feedback control strategy $\mathcal{U}(t, x, k) \in U_{\mathrm{CL}}$ such that all solutions of the differential inclusion (40) that start from any position $(t_0, x, k) \in T \times \mathcal{W}^*[t_0]$ would reach the set $\mathcal{M}$ regardless of disturbance $v(\cdot)$ restricted by (39).

This time it is assumed that the sections of target set $\mathcal{M}$ at constant levels of k are non-decreasing, i.e. when $k_1 \geqslant k_2$, then $\mathcal{M}(k_1) \supseteq \mathcal{M}(k_2)$.

Let us now briefly formulate main results concerning this problem.

5.1 *Solvability Domain*

Lemma 14. Minimax and maximin solvability domains within the class of open-loop control strategies can be found used the following relations for their sections at constant levels of k:

$$W^+[k, t] = W^+(k, t, t_1, \mathcal{M}(\cdot)) =$$
$$= \bigcup_{0 \leqslant \gamma \leqslant k} \left[\mathcal{M}(\gamma) - \sqrt{k - \gamma} \mathcal{P}(t, t_1) \right] \dot{-}$$
$$\dot{-} \int_t^{t_1} \mathcal{Q}(\tau) \, \mathrm{d}\tau, \tag{42}$$

$$W^-[k, t] = \bigcup_{0 \leqslant \gamma \leqslant k} \mathcal{M}(\gamma) \dot{-} \int_t^{t_1} \mathcal{Q}(\tau) \, \mathrm{d}\tau -$$
$$- \sqrt{k - \gamma} \mathcal{P}(t, t_1) \tag{43}$$

where $\mathcal{P}(t,t_1)$ is an ellipsoid

$$\mathcal{P}(t,t_1) = \mathcal{E}\left(0, \int_t^{t_1} P^{-1}(\tau)\,\mathrm{d}\tau\right). \qquad (44)$$

Upper and lower alternated integral are constructed the same way as in problem 2 using formulae (42) and (43).

Theorem 15. (1) Solvability set $\mathcal{W}^*[k,t]$ coincides with the alternated integral if the latter exists:

$$\mathcal{W}^*[k,t] = \mathcal{I}(k,t,t_1,\mathcal{M}(\cdot)). \qquad (45)$$

(2) Semigroup property in backward time (the principle of optimality) is true for the solvability set.

(3) Set-valued mapping $\mathcal{W}^*[k,t]$ is the unique solution of the evolution equation

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\Bigg(\mathcal{I}[k,t-\sigma]+\sigma\mathcal{Q}(t), \bigcup_{0\leqslant\gamma\leqslant k}\Big[\mathcal{I}[\gamma,t] - \sqrt{\sigma(k-\gamma)}\mathcal{E}\left(0,P^{-1}(t)\right)\Big]\Bigg) = 0. \qquad (46)$$

5.2 *Control Synthesis*

Consider the following value function:

$$V(t,x,k) = \min_{\substack{\mathcal{U}\in U_{\mathrm{CL}}\\ k(t)\geqslant 0}} \max_{\substack{v(\cdot)\\ v(\tau)\in\mathcal{Q}(\tau)}} \max_{(x[t_1],k[t_1])\in X[t_1]} d^2(x[t_1],\mathcal{M}(k[t_1])), \qquad (47)$$

where $X[t]$ is the attainability tube of the differential inclusion (40).

Theorem 16. (1) Value function (47) is the solution of H-J-B-I equation

$$\frac{\partial V}{\partial t} + \min_{u\in\mathbb{R}^n}\max_{v\in\mathcal{Q}(t)}\left\{\left\langle\frac{\partial V}{\partial x}, u+v\right\rangle - \langle u, P(t)u\rangle\frac{\partial V}{\partial k}\right\} = 0 \qquad (48)$$

with boundary condition

$$\left.\frac{\partial V}{\partial t} + \max_{v\in\mathcal{Q}(t)}\left\langle\frac{\partial V}{\partial x}, v\right\rangle\right|_{k=0} = 0 \qquad (49)$$

and initial condition

$$V(t_1,x,k) = d^2(x,\mathcal{M}(k)). \qquad (50)$$

(2) Solvability domain is the level set of value function:

$$\mathcal{W}^*[k,t] = \{x \mid V(t,x,k)\leqslant 0\}. \qquad (51)$$

(3) Value function is the distance to the solvability set:

$$V(t,x,k) = d^2(x,\mathcal{W}^*[k,t]), \qquad (52)$$

which means that guaranteed distance between trajectory endpoint and target set is the same as distance between initial point and solvability set. Formulae (51) and (52) establish a one-to-one relation between value functions and solvability domains.

(4) Value function satisfies the principle of optimality

$$V(t,x,k;V(t_1,\cdot,\cdot)) = V(t,x,k;V(\tau,\cdot,\cdot;V(t_1,\cdot,\cdot))) \qquad (53)$$

where $V(t_1,x,k) = d^2(x,\mathcal{M}(k))$.

(5) The following control strategy is optimal and it solves the problem 3:

$$\mathcal{U}^*(t,x,k) = \operatorname*{Arg\,min}_{u\in\mathbb{R}^n}\left\{\left\langle\frac{\partial V}{\partial x},u\right\rangle - \langle u,Pu\rangle\frac{\partial V}{\partial k}\right\} = \begin{cases}\mathbb{R}^n, & x\in\mathcal{W}^*[k,t],\\ \frac{1}{2}\left(\frac{\partial V}{\partial k}\right)^{-1}\frac{\partial V}{\partial x}, & x\notin\mathcal{W}^*[k,t].\end{cases} \qquad (54)$$

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