Hamiltonian Methods and Numerical Techniques for Closed-Loop Control of Oscillating Systems under Uncertainty

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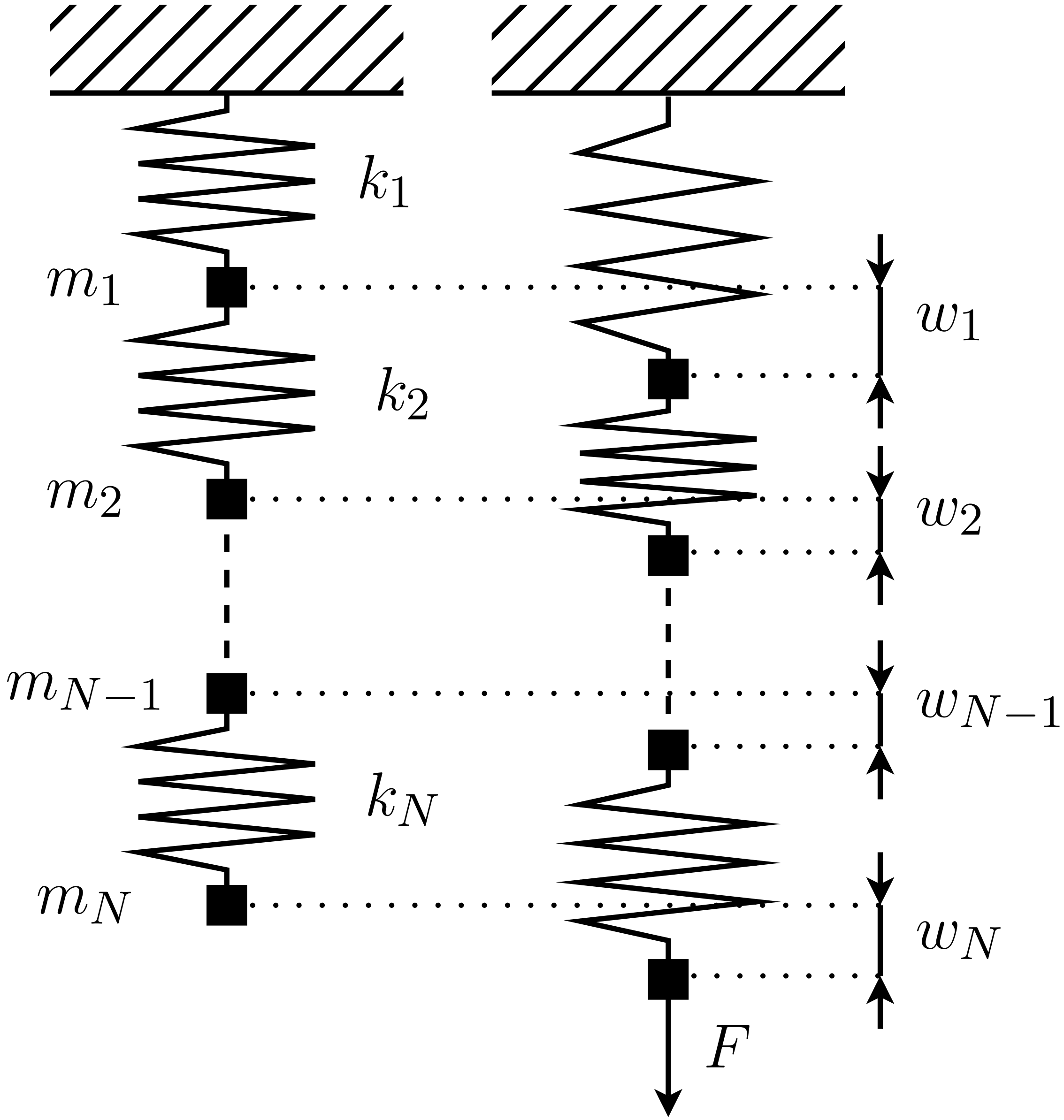
Oscillating System



Mathematical Model

$$\begin{cases} m_1 \ddot{w}_1(t) = k_2(w_2 - w_1) - k_1 w_1 + v_1(t) \\ m_i \ddot{w}_i(t) = k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}) + v_i(t) \quad i = \overline{2, N-1} \\ m_N \ddot{w}_N(t) = -k_N(w_N - w_{N-1}) + F(t) + v_N(t) \\ \\ w_i(t_0) = w_i^0 \quad \dot{w}_i(t_0) = \dot{w}_i^0 \quad i = \overline{1, N} \end{cases}$$

- w_i — displacements from the equilibrium
- F — control force
- v_i — unknown disturbing forces
- m_i — masses of the loads
- k_i — stiffness coefficients

$N \to \infty$

$$\rho(\xi) w_{tt}(t,\xi) = \big[Y(\xi) w_\xi(t,\xi)\big]_\xi, \quad t > t_0, \quad 0 < \xi < L$$

$$w(t,0) = 0, \quad w_\xi(t,L) = F(t)/Y(L), \quad t \geqslant t_0$$

$$w(t_0,\xi) = w^0(\xi), \quad w_t(t_0,\xi) = \dot{w}^0(\xi), \quad 0 \leqslant \xi \leqslant L$$

- $w(t,\xi)$ — displacement from the equilibrium
- $F(t)$ — control force
- $\rho(\xi)$ — mass density
- $Y(\xi)$ — Young modulus

Control and Disturbances

It is required to use continuous or smooth controls.

Control force is produced by an integrator

$$F(t) = \underbrace{\int_{t_0}^{t} \int_{t_0}^{\tau_\nu} \cdots \int_{t_0}^{\tau_2}}_{\nu \text{ times}} u(\tau_1)\,\mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\nu$$

$u(t)$ is the new control variable.

Hard bounds (geometrical constraints) on control and disturbance:

$$u(t) \in \mathscr{P} = [-\mu, \mu] \qquad v(t) \in \mathscr{Q} \subset \mathbb{R}^N$$

Normalized Matrix Form

$$\dot{x}(t) = Ax(t) + bu(t) + Cv(t)$$

$$x(t) = \begin{pmatrix} w(t) \\ \dot{w}(t) \\ F(t) \\ \dot{F}(t) \\ \vdots \\ F^{(\nu-1)}(t) \end{pmatrix} \quad w(t) = \begin{pmatrix} w_1(t) \\ \vdots \\ w_N(t) \end{pmatrix} \quad v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_N(t) \end{pmatrix}$$

Here $F^{(\nu)}(t) = u(t)$.

This system is **completely controllable**.

The Problem

Classes of controls:
Open-loop $\mathscr{U}_{\text{OL}}$: $u(t) : [t_0, t_1] \to \mathscr{P}$.
Closed-loop (feedback) $\mathscr{U}_{\text{CL}}$: $\mathscr{U}(t, x) : [t_0, t_1] \times \mathbb{R}^n \to \operatorname{conv} \mathscr{P}$.

The closed-loop system is a differential inclusion

$$\dot{x}(t) \in Ax(t) + b\mathscr{U}(t, x) + C\mathscr{Q}$$

Problem. For a given $\varepsilon > 0$, find

- *backward reach set (solvability domain)* $\mathscr{W}[t_0] \subseteq \mathbb{R}^{2N}$
- closed-loop control $\mathscr{U}(t, x)$

s.t. all trajectories starting in $\mathscr{W}[t_0]$ satisfy $\|x(t_1)\| \leqslant \varepsilon$.

Dynamic Programming Approach

The **value function**:

$$V(t,x) = \min_{\mathscr{U} \in \mathscr{U}_{\mathsf{CL}}} \max_{x(\cdot)} \left\{ \max\{0, \|x(t_1)\| - \varepsilon\}^2 \;\middle|\; x(t) = x \right\}$$

$$V(t,x) = V(t,x;t_1,V(t_1,\cdot)), \quad V(t_1,x) = \max\{0, \|x\| - \varepsilon\}^2$$

Principle of optimality:

$$V(t,x;t_1,V(t_1,\cdot)) = V(t,x;\tau,V(\tau,\cdot;t_1,V(t_1,\cdot))) \quad t \leqslant \tau \leqslant t_1$$

HJB Equation

Without disturbances ($\mathscr{Q} = \{0\}$):

$$V_t + \min_{|u| \leqslant \mu} \langle V_x, Ax + bu \rangle = 0 \quad t < t_1$$

$$V(t_1, x) = V(t_1, x) = \max \{0, \|x\| - \varepsilon\}^2$$

$$V(t, x) = d^2(\mathrm{e}^{(t_1 - t)A} x, \mathrm{e}^{(t_1 - t)A} \mathscr{W}[t])$$ (d is Euclidian distance)

$$\mathscr{W}[t] = \left\{ x \in \mathbb{R}^{2N+\nu} \;\middle|\; V(t, x) \leqslant 0 \right\}$$

$$\mathscr{U}^*(t, x) = \operatorname*{Arg\,min}_{|u| \leqslant \mu} \langle V_x, bu \rangle = \begin{cases} -\mu, & V_{x_{2N+\nu}} > 0; \\ \mu, & V_{x_{2N+\nu}} < 0; \\ [-\mu, \mu], & V_{x_{2N+\nu}} = 0. \end{cases}$$

HJBI Equation

$$V_t + \min_{|u|\leqslant\mu} \max_{v\in\mathscr{Q}} \langle V_x, Ax + bu + Cv\rangle = 0 \quad t < t_1$$

$$V(t_1, x) = V(t_1, x) = \max\{0, \|x\| - \varepsilon\}^2$$

$$V(t, x) \leqslant d^2(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t])$$

$$\mathscr{W}[t] = \left\{ x \in \mathbb{R}^{2N+\nu} \;\middle|\; V(t, x) \leqslant 0 \right\}$$

$$\mathscr{U}^*(t, x) = \operatorname*{Arg\,min}_{|u|\leqslant\mu} \langle V_x, bu\rangle = \begin{cases} -\mu, & V_{x_{2N+\nu}} > 0; \\ \mu, & V_{x_{2N+\nu}} < 0; \\ [-\mu, \mu], & V_{x_{2N+\nu}} = 0. \end{cases}$$

“Aiming” at $\mathscr{W}[t]$

$$\mathscr{U}^*(t,x) = \begin{cases} -\mu, & \ell^0_{2N+\nu} > 0 \\ \mu, & \ell^0_{2N+\nu} < 0 \\ [-\mu,\mu], & \ell^0_{2N+\nu} = 0 \end{cases}$$

$$d^2(\mathrm{e}^{(t_1-t)A}x, \mathrm{e}^{(t_1-t)A}\mathscr{W}[t]) = \max_{\ell\in\mathbb{R}^n}\left\{\langle \ell, x\rangle - \rho(\,\ell \mid \mathscr{W}[t]) - \frac{1}{4}\left\|\mathrm{e}^{(t-t_1)A}\ell\right\|^2\right\} =$$

$$= \langle \ell^0, x\rangle - \rho\big(\,\ell^0 \mid \mathscr{W}[t]\big) - \frac{1}{4}\left\|\mathrm{e}^{(t-t_1)A}\ell^0\right\|^2$$

$\rho(\,\ell \mid \mathscr{W}[t]) = \max\limits_{x\in\mathscr{W}[t]} \langle \ell, \mathscr{W}[t]\rangle = \mu\int_t^{t_1} |s_{2N}(\tau)|\,\mathrm{d}\tau + \varepsilon\|\ell\|$ (without disturbances)

The adjoint equation $\dot{s}(t) = -A^T s(t)$, $s(t_1) = \ell$ or $s(t) = \mathrm{e}^{(t-t_1)A}\ell$

Approximation

The exact solvability domain $\mathscr{W}[t]$
is replaced by an approximation $\mathscr{Z}[t]$, such that

- the set $\mathscr{Z}[t]$ is described by a smaller amount of data
- "aiming" strategy at $\mathscr{Z}[t]$ is easily calculated and damps the system

Ellipsoids

$$\mathscr{E}(m, M) = \left\{ x \mid \left\langle x - m, M^{-1}(x - m) \right\rangle \leqslant 1 \right\}$$

$$\rho(\ell \mid \mathscr{E}(m, M)) = \langle m, \ell \rangle + \sqrt{\langle \ell, M\ell \rangle}$$

(A. B. Kurzhanski and I. Vályi, Ellipsoidal Calculus for Estimation and Control. Boston: Birkhäuser, 1997.)

Ellipsoidal Approximation

$$\mathscr{X}[t] = \mathscr{E}(x^*(t), X_-(t))$$

$$\dot{x}^*(t) = Ax^*(t) + Cq$$

$$\dot{X}_-(t) = AX_-(t) + X_-(t)A^T - \pi(t)X_-(t) - \pi^{-1}(t)CQC^T +$$

$$+X_-^{\frac{1}{2}}(t)S(t)(bPb^T)^{\frac{1}{2}} + (bPb^T)^{\frac{1}{2}}S^T(t)X_-^{\frac{1}{2}}(t)$$

$$x^*(t_1) = 0 \quad X_-(t) = 0$$

$$S(t)P^{\frac{1}{2}}B^Ts(t) = \lambda(t)X_-^{\frac{1}{2}}s(t) \quad S^T(t)S(t) = I \qquad \pi(t) = \frac{\left\langle s(t), CQC^Ts(t)\right\rangle^{\frac{1}{2}}}{\langle s(t), X_-(t)s(t)\rangle^{\frac{1}{2}}}$$

$$\mathscr{Q} = \mathscr{E}(q, Q)$$

$$\rho(\, s(t) \mid \mathscr{X}[t]) = \rho(\, s(t) \mid \mathscr{W}[t])$$

(A. B. Kurzhanski and P. Varaiya, "Ellipsoidal techniques for reachability analysis. Part II: Internal approximations. Box-valued constraints," Optimization methods and software, *vol. 17, pp. 177–237, 2002.)*

Evolution Equation

$$\lim_{\sigma\to 0+} \sigma^{-1} h_+(\mathscr{Z}[t-\sigma] + \sigma C\mathscr{Q}, (I-\sigma A)\mathscr{Z}[t] - \sigma b\mathscr{P}) = 0$$

$$\mathscr{Z}[t_1] = \mathscr{B}_\varepsilon(0)$$

$$h_+(\mathscr{X}, \mathscr{Y}) = \min\{r > 0 \mid \mathscr{X} \subseteq \mathscr{Y} + \mathscr{B}_r\}$$

Particular solutions:

- $\mathscr{W}[t]$ is the maximum solution w.r.t. inclusion
- Ellipsoidal approximations

“Aiming” at $\mathscr{Z}[t]$

$$Z(t,x) = d^2(x, \mathscr{Z}[t])$$

$$\min_{|u|\leqslant\mu} \max_{v\in\mathscr{Q}} \frac{\mathrm{d}Z(t,x(t))}{\mathrm{d}t} = Z_t + \min_{|u|\leqslant\mu} \max_{v\in\mathscr{Q}} \langle Z_x, Ax + bu + Cv\rangle \leqslant 0.$$

$$\mathscr{U}_{\mathscr{Z}}(t,x) = \operatorname*{Arg\,min}_{|u|\leqslant\mu} \langle Z_x, bu\rangle$$

If $x(t) \in \mathscr{Z}[t]$, then

$$\|x(t_1)\| = d(x(t_1), \mathscr{Z}[t_1]) \leqslant d(x(t), \mathscr{Z}[t]) = 0$$

Ellipsoidal Control Synthesis

$$\mathscr{U}_{\mathscr{X}}(t,x) = \begin{cases} -\mu, & \ell^0_{2N+\nu} > 0; \\ \mu, & \ell^0_{2N+\nu} < 0; \\ [-\mu, \mu], & \ell^0_{2N+\nu} = 0, \end{cases}$$

$$\ell^0 = X_-^{-1}[t] \cdot (x - x^*(t))$$

Ellipsoidal toolbox:

http://www.eecs.berkeley.edu/~akurzhan/ellipsoids/

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